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**ROBUST SOLUTIONS FOR SYSTEMS OF  
UNCERTAIN LINEAR EQUATIONS**

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# Robust Solutions for Systems of Uncertain Linear Equations

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Our contribution is twofold. Firstly, for a system of uncertain linear equations where the uncertainties are column-wise and reside in general convex sets, we show that the intersection of the set of possible solutions and any orthant is convex. We derive a convex representation of this intersection to calculate the ranges of the coordinates. Secondly, we propose two new methods for obtaining robust solutions of systems of uncertain linear equations. The first method calculates the center of the maximum inscribed ellipsoid of the set of possible solutions. The second method minimizes the expected violations with respect to the worst-case distribution. We compare these two new methods both theoretically and numerically with an existing method. The existing method minimizes the worst-case violation. Applications to the input-output model, Colley's Matrix Rankings and Article Influence Scores demonstrate the advantages of the two new methods.

*Key words:* Interval linear systems; uncertain linear equations; maximum volume inscribed ellipsoid; worst-case distribution; robust least-squares; input-output model; Colley's Matrix Rankings; Article Influence Scores.

*JEL codes:* C61

*History:*

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## 1. Introduction

Systems of linear equations are of immense importance in mathematics and its applications in physics, economics, engineering, and many more fields. However, the presence of unavoidable errors (inaccuracies) in the specification of parameters in both the right- and left-hand sides introduces uncertainty in the sought solution. The uncertainties may be raised due to measurement/rounding errors in the data of the physical problems, estimation errors in the estimated parameters by using expert opinions and/or historical data, or numerical errors associated with finite representation of numbers by computer (see Ben-Tal et al. (2009)).

A basic version of the problem that we consider in this paper is well known in the context of interval linear systems. For a given system of linear equations,

$$A\mathbf{x} = \mathbf{b}, \quad (1)$$

where the coefficient matrix  $A \in \mathbb{R}^{n \times n}$  and right-hand side  $\mathbf{b} \in \mathbb{R}^n$  are uncertain and allowed to vary uniformly and independently of each other in the given intervals,

$$\mathcal{U} = \{ (A, \mathbf{b}) : \underline{a}_{ij} \leq a_{ij} \leq \bar{a}_{ij}, \underline{b}_i \leq b_i \leq \bar{b}_i, \forall i, j \},$$

where  $\underline{a}_{ij}, \bar{a}_{ij}, \underline{b}_i, \bar{b}_i \in \mathbb{R}$ , for all  $i, j$ , are the lower- and upper-bounds of the components in the matrix  $A$  and vector  $\mathbf{b}$ , respectively. Each of the possible  $(A, \mathbf{b}) \in \mathcal{U}$  has equal claim to be the true realization of the physical problem. Different parameters may produce different solutions for the system of linear equations. The set  $\mathcal{X}$  of all feasible solutions of the system (1), where  $(A, \mathbf{b}) \in \mathcal{U}$ , is defined as:

$$\mathcal{X} = \{ \mathbf{x} \in \mathbb{R}^n \mid \exists (A, \mathbf{b}) \in \mathcal{U} : A\mathbf{x} = \mathbf{b} \}.$$

Other definitions of solution sets can be found in the books Kreinovich et al. (1998) and Shary (2011). Since the pioneer work by Oettli and Prager (1964), much literature has been devoted to describe the ranges of the components of the solution  $\mathbf{x} \in \mathcal{X}$  for interval linear systems, i.e.,

$$\begin{aligned} \bar{x}_i &= \max\{ x_i \in \mathbb{R} : \mathbf{x} \in \mathcal{X} \} \\ \underline{x}_i &= \min\{ x_i \in \mathbb{R} : \mathbf{x} \in \mathcal{X} \}, \end{aligned} \quad (2)$$

where  $x_i$  denotes the  $i$ -th element of vector  $\mathbf{x}$ . The main source of difficulties connected with obtaining ranges of  $x_i$  is the complicated structure of the solution set  $\mathcal{X}$ , which is generally non-convex. The intersection of the solution set and each orthant is, however, a convex polyhedron. Oettli (1965) proposes using a linear programming procedure in each orthant (i.e.,  $2^n$  orthants in total) to determine  $\bar{x}_i$  and  $\underline{x}_i$ . Rohn and Kreinovich (1995) show that, in general, determining the exact ranges for the components of  $\mathbf{x} \in \mathcal{X}$  for an interval linear system is an NP-hard problem. For a comprehensive treatment and for references to the literature on interval linear systems one may refer to the books Neumaier (1990), Kreinovich et al. (1998), Fiedler et al. (2006), and Moore et al. (2009). Due to the NP-hardness of solving (2) exactly, many ingenious methods have been developed to obtain sufficiently close outer estimates of the solution set  $\mathcal{X}$ . We refrain here from listing papers dedicated to computing enclosures since they are simply too many. Hansen (1992) introduces six interrelated methods to sharpen the bounds of the estimated solution set for interval linear systems. In the situations which the solution set intersects only few orthants, the algorithm of Jansson (1997) can be applied. Calafiore and El Ghaoui (2004) consider a more general situation

(than interval uncertainties) in which the coefficient matrix  $A$  and vector  $\mathbf{b}$  belong to an uncertainty set  $\mathcal{U}$  described by means of a linear fractional representation. They propose a method that finds the ellipsoidal outer bounds of the solution set of a system of uncertain linear equations by using semidefinite programming. Rohn (1981) and Alefeld et al. (1998) handle interval linear systems with dependent data. Many other methods can also be found in Neumaier (1990) and Fiedler et al. (2006). Interval linear systems has been applied to many engineering problems described by systems of linear equations involving uncertainties. These problems include analysis of mechanical structures (see Smith et al. (2012), Muhanna and Erdolen (2006)), electrical circuit designs (see Dreyer (2005), Kolev (1993)) and chemical engineering (see Gau and Stadtherr (2002)). For more applications we refer to the book Moore et al. (2009).

In this paper, we consider the system of uncertain linear equations:

$$A(\boldsymbol{\zeta})\mathbf{x} = \mathbf{b}(\boldsymbol{\zeta}), \quad (3)$$

where the coefficient matrix  $A: \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$  and right-hand side  $\mathbf{b}: \mathbb{R}^m \rightarrow \mathbb{R}^n$  are affine in  $\boldsymbol{\zeta} \in \mathbb{R}^m$ , and the uncertain parameter  $\boldsymbol{\zeta}$  resides in the uncertainty set  $\mathcal{U}$ . Firstly, we focus on systems of linear equations with column-wise uncertainties. Let  $A(\boldsymbol{\zeta}) = [\mathbf{a}_{\cdot 1}(\boldsymbol{\zeta}_1) \ \mathbf{a}_{\cdot 2}(\boldsymbol{\zeta}_2) \ \cdots \ \mathbf{a}_{\cdot n}(\boldsymbol{\zeta}_n)]$ ,  $\mathbf{b}(\boldsymbol{\zeta}) = \mathbf{b}(\boldsymbol{\zeta}_0)$ , and  $\boldsymbol{\zeta} = [\boldsymbol{\zeta}_0^T \ \boldsymbol{\zeta}_1^T \ \cdots \ \boldsymbol{\zeta}_n^T]^T$ . We represent the system (3) as follows:

$$\sum_{j=1}^n \mathbf{a}_{\cdot j}(\boldsymbol{\zeta}_j) x_j = \mathbf{b}(\boldsymbol{\zeta}_0), \quad (4)$$

where the components of the vector  $\mathbf{a}_{\cdot j}$  is affine in  $\boldsymbol{\zeta}_j \in \mathcal{U}_j$ , vector  $\mathbf{b}$  is affine in  $\boldsymbol{\zeta}_0 \in \mathcal{U}_0$ , and the set  $\mathcal{U}_j$  is convex, for all  $j = 0, 1, \dots, n$ . The corresponding solution set  $\mathcal{X}$  is:

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \forall j, \exists \boldsymbol{\zeta}_j \in \mathcal{U}_j : \sum_{j=1}^n \mathbf{a}_{\cdot j}(\boldsymbol{\zeta}_j) x_j = \mathbf{b}(\boldsymbol{\zeta}_0) \right\}. \quad (5)$$

Frequently in practice, due to physical or economic properties, the solutions do not change their signs. We assume that the signs of the solutions  $\mathbf{x} \in \mathcal{X}$  are predetermined. Oettli (1965) shows that for interval uncertainties, the intersection of the solution set  $\mathcal{X}$  and any orthant of  $\mathbb{R}^n$  is a convex polyhedron. We generalize the results for interval linear systems. For column-wise (dependent) uncertainties that reside in convex sets, we show that the solution set  $\mathcal{X}$  in any orthant is convex. Moreover, when  $\mathcal{U}_j$  is polyhedral, the solution set  $\mathcal{X}$  is also polyhedral; when  $\mathcal{U}_j$  is ellipsoidal, the solution set  $\mathcal{X}$  is conic quadratic representable. Then, we derive a convex representation of  $\mathcal{X}$  to calculate the exact range of  $x_j$ , for all  $j$ , within a specific orthant.

Secondly, we propose two new ways for computing robust solutions of systems of uncertain linear equations that are uncertain. The first robust solution  $\mathbf{x}_{MVE}$  is defined as the (unique) center of the maximum volume inscribed ellipsoid (abbr., MVE) of a polyhedral solution set  $\mathcal{X}$ . It is intuitively

appealing to find a centralized solution that is “far” from the boundaries of the solution set  $\mathcal{X}$  (i.e., infeasibility). The MVE center  $\mathbf{x}_{MVE}$  is, by definition, in the (relative) interior of  $\mathcal{X}$  and it is *affine invariant*. Since for a general convex set, finding the MVE can be computationally intractable, for this method, we focus on polyhedral  $\mathcal{U}_j$  for all  $j$ . We apply the method developed in Zhen and den Hertog (2015) to compute the MVE center of the solution set  $\mathcal{X}$ .

In the second method, we determine the minimizer  $\mathbf{x}_{(\mu, \mathbf{d})}$  of the expected violations of the system (3) with respect to the worst-case distribution of  $\boldsymbol{\zeta} \in \mathbb{R}^m$ :

$$(EWD) \quad \min_{\mathbf{x}} \sup_{\mathbb{P}_{\boldsymbol{\zeta}} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} [\|A(\boldsymbol{\zeta})\mathbf{x} - \mathbf{b}(\boldsymbol{\zeta})\|_2^2],$$

where  $\|\cdot\|_2$  is the 2-norm, and the uncertainty set  $\mathcal{P}$  is a set of probability distributions that satisfy the following partial information about  $\boldsymbol{\zeta}$ : the support-including interval, mean  $\boldsymbol{\mu}$  and mean absolute deviation  $\mathbf{d}$ . The components of  $\boldsymbol{\zeta}$  are independently distributed. We apply the results in Ben-Tal and Hochman (1972) to derive a closed-form expression for

$$(EP) \quad \sup_{\mathbb{P}_{\boldsymbol{\zeta}} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}} [\|A(\boldsymbol{\zeta})\mathbf{x} - \mathbf{b}(\boldsymbol{\zeta})\|_2^2].$$

The resulting closed-form expression is a quadratic function in the (only remaining) variable  $\mathbf{x}$ . Then, the min-max problem (EWD) becomes a quadratic programming (QP) problem. We also consider a simpler case where only the support-including intervals and means about  $\boldsymbol{\zeta}$  are provided, and apply the results in Madansky (1959) to solve (EWD). Postek et al. (2015) is the first paper that applies the results in Madansky (1959) and Ben-Tal and Hochman (1972) to Robust Optimization. A more complicated but general framework for optimization with respect to the worst-case distribution is introduced in Wiesemann et al. (2014). Since their approach can only be applied to special classes of functions and it cannot handle independency among the components of  $\boldsymbol{\zeta}$  (see Hanasusanto et al. (2015)), we do not consider it in the present paper.

We compare these two new methods both theoretically and numerically with an existing method. The conventional approach of determining a robust solution for systems of uncertain linear equations first appears in the context of robust least-squares (RLS) problems El Ghaoui and Lebret (1997). The RLS method finds the minimizer  $\mathbf{x}_{RLS}$  of the worst-case 2-norm violation of the system:

$$(RLS) \quad \min_{\mathbf{x}} \max_{\boldsymbol{\zeta} \in \mathcal{U}} \|A(\boldsymbol{\zeta})\mathbf{x} - \mathbf{b}(\boldsymbol{\zeta})\|_2.$$

The tractability of the problem (RLS) strongly relies on the choice of the uncertainty set  $\mathcal{U}$ . In this paper, we focus on independent interval uncertainties for the problem (RLS):

$$\mathcal{U} = \bigcup_{j=0}^n \mathcal{U}_j, \quad \text{where } \mathcal{U}_j = \{ \boldsymbol{\zeta}_j \in \mathbb{R}^n : \underline{\zeta}_j \leq \zeta_j \leq \bar{\zeta}_j \}, \quad \forall j = 0, 1, \dots, n.$$

Each component of  $A$  and  $\mathbf{b}$  resides in an independent interval:

$$A(\boldsymbol{\zeta}) = [\zeta_1 \ \zeta_2 \ \cdots \ \zeta_n], \quad \mathbf{b}(\boldsymbol{\zeta}) = \zeta_0$$

where  $\boldsymbol{\zeta} = [\zeta_0^T \ \zeta_1^T \ \cdots \ \zeta_n^T]^T \in \mathbb{R}^m$ . Ben-Tal et al. (2009) show that problem (*RLS*) under independent interval uncertainties can be reformulated into an SOCP problem. In El Ghaoui and Lebret (1997), Beck and Eldar (2006) and Jeyakumar and Li (2014), authors derive an SOCP or a semidefinite programming (SDP) reformulation of the problem (*RLS*) under ellipsoidal uncertainties. Burer (2012) and Juditsky and Polyak (2012) solve (*RLS*) to find the robust rating vectors for Colley's Matrix Ranking and Google's PageRank, respectively.

The contributions of this paper may be summarized as follows:

1. We generalize the results for interval linear systems. For column-wise (dependent) uncertainties that reside in convex sets, we show that the solution set  $\mathcal{X}$  in any orthant is convex. We derive a convex representation of  $\mathcal{X}$  to calculate the exact range of  $x_j$ , for all  $j$ , within a specific orthant.
2. We introduce two new ways for obtaining the robust solutions of systems of uncertain linear equations. The first method finds a centralized solution in the solution set. The second method minimizes the expected violation with respect to the worst-case distribution of the uncertain parameters.
3. We compare the two new methods both theoretically and numerically with the RLS method. We show that the robust solutions  $\mathbf{x}_{(\mu, \mathbf{d})}$  and  $\mathbf{x}_{RLS}$  are scale sensitive and may even be outside the solution set  $\mathcal{X}$ . The robust solutions  $\mathbf{x}_{MVE}$ ,  $\mathbf{x}_{(\mu, \mathbf{d})}$  and  $\mathbf{x}_{RLS}$  can be obtained by solving an SDP, QP or an SOCP problem, respectively. Applications to the input-output model, Colley's Matrix Rankings, and Article Influence Scores demonstrate the advantages and disadvantages of the three methods.

The remainder of the paper is organized as follows. §2 first discusses the properties of the solution set  $\mathcal{X}$ . Then, we derive an equivalent convex representation of  $\mathcal{X}$ . §3 discusses the method for computing the MVE center  $\mathbf{x}_{MVE}$  in  $\mathcal{X}$ . §4 presents the method to solve problem (*EWD*), when partial distributional information about the uncertain parameters is available. In §5, we theoretically compare the two new robust solution methods with the RLS method, and §6 presents numerical results.

**Notation** Here we briefly introduce our notations. We use bold faced characters such as  $\mathbf{x} \in \mathbb{R}^n$  to represent vectors. We use  $x_i$  to denote the  $i$ -th element of the vector  $\mathbf{x}$ . We denote  $\mathbf{a}_{.j}$  as the  $j$ -th column of the matrix  $A$ . We use normal and mathematical capital letters such as  $A \in \mathbb{R}^{n \times n}$  and  $\mathcal{X}$  to represent matrices and sets, respectively. We denote  $\boldsymbol{\zeta} \in \mathbb{R}^m$  as the uncertain parameter. We denote  $\mathcal{P}$  as the set of probability distributions. Given a random variable  $\boldsymbol{\zeta} \in \mathbb{R}^m$  with

probability distribution  $\mathbb{P}_\zeta \in \mathcal{P}$  and a function  $g: \mathbb{R}^m \rightarrow \mathbb{R}$ , we denote  $\mathbb{E}_{\mathbb{P}_\zeta} [g(\zeta)]$  as the expectation of the random variable  $g(\zeta)$  over the probability distribution  $\mathbb{P}_\zeta$ .

## 2. Properties and Representation of the Solution Set

In this section, we first present the properties of the solution set. We show that, for column-wise (dependent) uncertainties in general convex sets, the solution set in any orthant is convex. In §2.2, we derive a convex representation of the solution set  $\mathcal{X}$ .

### 2.1. Properties of the Solution Set

Let us consider system (4) of uncertain linear equations. We assume that the uncertainty set  $\mathcal{U}_j$  that  $\zeta_j$  resides in, is bounded and defined as follows:

$$\mathcal{U}_j = \{\zeta_j \mid \forall k: f_{jk}(\zeta_j) \leq 0\}, \quad \forall j = 0, 1, \dots, n, \quad (6)$$

and the function  $f_{jk}$  is convex in  $\zeta_j$ , for all  $j$  and  $k$ . The components of  $\zeta_j \in \mathcal{U}_j$  may be dependent. For  $i \neq j$ , the components of  $\zeta_i \in \mathcal{U}_i$  and  $\zeta_j \in \mathcal{U}_j$  are independent. The uncertainties in the system (4) are indeed column-wise. Note that the dimensions of the uncertain parameters  $\zeta_i$  and  $\zeta_j$  are not necessarily the same for  $i \neq j$ . The following theorem shows that the intersection of the solution set of system (4) with uncertainty sets  $\mathcal{U}_j$  defined in (6) and the non-negative orthant, i.e.,  $\mathbb{R}_+^n$ , is convex.

**THEOREM 1.** *For the uncertainty sets defined in (6), the intersection of the solution set  $\mathcal{X}$  and the non-negative orthant*

$$\mathcal{X} \cap \mathbb{R}_+^n = \left\{ \mathbf{x} \in \mathbb{R}_+^n \mid \forall j, \exists \zeta_j \in \mathcal{U}_j: \sum_{j=1}^n \mathbf{a}_{\cdot j}(\zeta_j) x_j = \mathbf{b}(\zeta_0) \right\}$$

*is convex.*

*Proof.* It follows from the proof of Blanc and Hertog (2008, Proposition 1). Q.E.D.

In fact, it can be shown that the intersection of the solution set  $\mathcal{X}$  and **any** orthant of  $\mathbb{R}^n$  is convex. Since the proof is almost identical, we do not include it in this paper. Example 1 and 2 show that in order to preserve convexity of the solution set  $\mathcal{X}$  the following two conditions are necessary. Firstly, the feasible solutions  $\mathbf{x} \in \mathcal{X}$  are within a particular orthant of  $\mathbb{R}^n$ . Secondly, the uncertainties in the matrix  $A(\zeta)$  and vector  $\mathbf{b}(\zeta)$  are column-wise.

**EXAMPLE 1. The union of the solutions  $\mathbf{x} \in \mathcal{X}$  in different orthants can be nonconvex.**

*Let us consider the following solution set of a given uncertain linear system*

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^2 : \begin{bmatrix} \zeta_1 & 0 \\ \zeta_1 & \zeta_2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \zeta_1 \in [-1, 1], \zeta_2 \in [1, 2] \right\}.$$



The solution set  $\mathcal{X}$  can be represented as:

$$\mathcal{X} = \left\{ \mathbf{x} \in \mathbb{R}^2 : x_1 \in (-\infty, -1] \cup [1, +\infty), x_2 \in [-1, -\frac{1}{2}] \right\}.$$

One can easily see that the intersection of the solution set  $\mathcal{X}$  and each orthant of  $\mathbb{R}^2$  is indeed convex. However, the set  $\mathcal{X}$  is nonconvex.

Alefeld et al. (1998) show that, if the uncertainties in the system (4) are not column-wise (e.g.,  $A(\zeta)$  is symmetric), the solution set  $\mathcal{X}$  may be nonconvex. This is illustrated by Example 2.

**EXAMPLE 2. The solution set  $\mathcal{X}$  can be nonconvex when the uncertainties are not column-wise.** Let us consider the following solution set of an uncertain linear system in  $\mathbb{R}_+^2$ :

$$\mathcal{X} \cap \mathbb{R}_+^2 = \left\{ \mathbf{x} \in \mathbb{R}_+^2 : \begin{bmatrix} \zeta_1 & 0 \\ \zeta_1 & -\zeta_1 - \zeta_2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \zeta_1 \in [1, 2], \zeta_2 \in [-1, 1] \right\}.$$

Note that the uncertainties of the system are not column-wise. The set can be represented as:

$$\mathcal{X} \cap \mathbb{R}_+^2 = \left\{ \mathbf{x} \in \mathbb{R}_+^2 : |x_1 - x_2| - x_1 x_2 \leq 0, \frac{1}{2} \leq x_1 \leq 1 \right\}.$$

Clearly, the set  $\mathcal{X} \cap \mathbb{R}_+^2$  is nonconvex.

## 2.2. Convex Representation of the Feasible Solution Set

Given the uncertainty sets defined in (6), the intersection of the solution set  $\mathcal{X}$  and the non-negative orthant  $\mathbb{R}_+^n$  can be compactly represented as follows:

$$\mathcal{X} \cap \mathbb{R}_+^n = \left\{ \mathbf{x} \in \mathbb{R}_+^n \mid \forall j, k: \sum_{j=1}^n \mathbf{a}_{\cdot j}(\zeta_j) x_j = \mathbf{b}(\zeta_0), f_{jk}(\zeta_j) \leq 0, f_{0k}(\zeta_0) \leq 0 \right\}, \quad (7)$$

where the components of the vectors  $\mathbf{a}_{\cdot j}(\zeta_j), \mathbf{b}(\zeta_0) \in \mathbb{R}^n$  are affine in  $\zeta_j$ , and  $f_{jk}$  is convex in  $\zeta_j$  for all  $j, k = 1, 2, \dots, n$ . Due to the presence of products of variables (e.g.,  $\zeta_j x_j$  for some  $j$ ), the representation of set (7) is nonconvex.

A equivalent convex representation of the set (7) can be obtained by substituting  $\mathbf{y}_j = x_j \zeta_j$  and multiply the inequality constraints containing  $\zeta_j$  with  $x_j$ :

$$\mathcal{X} \cap \mathbb{R}_+^n = \left\{ \mathbf{x} \in \mathbb{R}_+^n \mid \forall j, k: \exists \mathbf{y}_j, \zeta_0: \sum_{j=1}^n \mathbf{a}_{\cdot j} \left( \frac{\mathbf{y}_j}{x_j} \right) x_j = \mathbf{b}(\zeta_0), x_j f_{jk} \left( \frac{\mathbf{y}_j}{x_j} \right) \leq 0, f_{0k}(\zeta_0) \leq 0 \right\}, \quad (8)$$

where  $0\mathbf{a}_{\cdot j} \left( \frac{\mathbf{y}_j}{0} \right) = \lim_{x_j \rightarrow 0} x_j \mathbf{a}_{\cdot j} \left( \frac{\mathbf{y}_j}{x_j} \right)$  and  $0f_{jk} \left( \frac{\mathbf{y}_j}{0} \right) = \lim_{x_j \rightarrow 0} x_j f_{jk} \left( \frac{\mathbf{y}_j}{x_j} \right)$  are the recession functions of  $\mathbf{a}_{\cdot j}$  and  $f_{jk}$ , respectively (see Rockafellar (1997)). Dacorogna and Maréchal (2008) show that, for a convex function  $f_{jk}$ , its perspective  $g_{jk}(\mathbf{y}_j, x_j) := x_j f_{jk} \left( \frac{\mathbf{y}_j}{x_j} \right)$  is convex on  $\mathbb{R}^n \times \mathbb{R}_+$ . Hence,  $\mathcal{X}$  is convex. Moreover, for all  $j, k$ , if  $f_{jk}$  is affine in  $\zeta_j$ , the set  $\mathcal{X}$  is polyhedral; if  $f_{jk}$  is quadratic in

$\zeta_j$ , the set  $\mathcal{X}$  is conic quadratic representable. In fact, for general convex functions  $f_{jk}$ , for all  $j, k$ , the solution set  $\mathcal{X}$  in any orthant of  $\mathbb{R}^n$  is convex, which coincides with the findings in Theorem 1. Our result generalizes the results of Oettli (1965), where the author shows that for interval uncertainties, the intersection of the solution set  $\mathcal{X}$  and any orthant of  $\mathbb{R}^n$  is polyhedral. Moreover, we provide a convex representation of  $\mathcal{X}$ , which can be used to calculate the exact range of  $x_j$ , for all  $j$ , within a specific orthant.

This transformation technique is first proposed in Dantzig (1963) to solve Generalized LPs. It is also applied to the dual of LPs with polyhedral uncertainty in Römer (2010). Gorissen et al. (2014) use this technique to derive tractable robust counterparts of a linear conic optimization problem. We illustrate this transformation by the following interval linear system example. This example is used throughout this paper.

**EXAMPLE 3. The convex representation of  $\mathcal{X} \cap \mathbb{R}_+^n$  with product of variables** *Let us consider the solution set in non-negative orthant:*

$$\mathcal{X} \cap \mathbb{R}_+^2 = \{ \mathbf{x} \in \mathbb{R}_+^2 \mid \forall j, \exists \zeta_j \in \mathcal{U}_j : \zeta_1 x_1 + \zeta_2 x_2 = \zeta_0 \},$$

where  $\mathcal{U}_0 = [0, 120] \times [60, 240]$ ,  $\mathcal{U}_1 = [0, 1] \times \{2\}$  and  $\mathcal{U}_2 = [2, 3] \times [1, 2]$ . Substituting  $\zeta_1 = \frac{y_1}{x_1}$  and  $\zeta_2 = \frac{y_2}{x_2}$ , and multiplying the inequality constraints containing  $\zeta_j$  with  $x_j$ , yields the following representation:

$$\mathcal{X} \cap \mathbb{R}_+^2 = \left\{ \mathbf{x} \in \mathbb{R}_+^2 \mid \exists \mathbf{y}_1, \mathbf{y}_2, \zeta_0 : \begin{array}{l} \mathbf{y}_1 + \mathbf{y}_2 = \zeta_0, \quad \begin{bmatrix} 0 \\ 60 \end{bmatrix} \leq \zeta_0 \leq \begin{bmatrix} 120 \\ 240 \end{bmatrix} \\ \begin{bmatrix} 0 \\ 2x_1 \end{bmatrix} \leq \mathbf{y}_1 \leq \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix}, \quad \begin{bmatrix} 2x_2 \\ x_2 \end{bmatrix} \leq \mathbf{y}_2 \leq \begin{bmatrix} 3x_2 \\ 2x_2 \end{bmatrix} \end{array} \right\}. \quad (9)$$

One can further simplify this set by eliminating the equality constraints. From Figure 1, we observe that the set defined in (9) is a full-dimensional polytope.

For interval linear systems, Kreinovich et al. (1998) show that checking the boundedness of the solution set is NP-hard. If we only focus on the solution set in a specific orthant, the boundedness can be checked in polynomial time. E.g., the boundedness of the set (8) can be checked by maximizing  $\sum_{i=1}^n x_i$  over  $\mathcal{X}$  in  $\mathbb{R}_+^n$ .

### 3. MVE Center of the Solution Set

Firstly, in §3.1, we present the method of Zhen and den Hertog (2015) for computing the MVE center of a polytope with respect to a subset of variables. Since the obtained MVE is an under approximation of the optimal MVE, in §3.2, we briefly discuss a simple procedure to provide an upper approximation.

### 3.1. Computing MVE Center in a Polytopic Projection

It is well-known that for a general convex set, finding the MVE can be computationally intractable. In this subsection, we focus on a polyhedral set. The set defined in (8) is polyhedral if the functions  $f_{jk}$  and  $f_{0k}$  are affine, for all  $j, k$ :

$$\mathcal{X} \cap \mathbb{R}_+^n = \left\{ \mathbf{x} \in \mathbb{R}_+^n \mid \forall j, k \exists \mathbf{y}_j, \boldsymbol{\zeta}_0 : \begin{array}{l} \sum_{j=1}^n \mathbf{a}_{\cdot j} \left( \frac{\mathbf{y}_j}{x_j} \right) x_j = \mathbf{b}(\boldsymbol{\zeta}_0), \\ c_{jk}x_j + \mathbf{d}_{jk}^T \mathbf{y}_j \leq 0, c_{0k} + \mathbf{d}_{0k}^T \boldsymbol{\zeta}_0 \leq 0 \end{array} \right\}, \quad (10)$$

where  $c_{jk}, c_{0k} \in \mathbb{R}$ ,  $\mathbf{d}_{jk}, \mathbf{d}_{0k} \in \mathbb{R}^n$ . For the sake of convenience, we represent the polyhedral set (10) as follows:

$$\mathcal{H} = \left\{ \mathbf{x} \in \mathbb{R}^n \mid \exists \mathbf{y} : D \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \mathbf{c} \right\}, \quad (11)$$

where  $D \in \mathbb{R}^{l \times t}$ ,  $\mathbf{c} \in \mathbb{R}^l$ ,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{y} \in \mathbb{R}^s$ , and  $t = s + n$  for some  $l, s \in \mathbb{R}$ . The auxiliary variable  $\mathbf{y}$  in (11) represents the variables  $\mathbf{y}_j$ 's and  $\boldsymbol{\zeta}_0$  in (10).

Boyd and Vandenberghe (2004) compute the MVE center with respect to **all** the variables of a full-dimensional polytope by solving an SDP problem. We refer to this method as the classical method. The description of  $\mathcal{H}$  contains the main variable  $\mathbf{x}$  and auxiliary variable  $\mathbf{y}$ . We are interested in the MVE center **only** with respect to the variable  $\mathbf{x}$ . Because of the existence of the auxiliary variable  $\mathbf{y}$  in the description of  $\mathcal{H}$ , the classical method described in Boyd and Vandenberghe (2004) may be too restrictive. One can use elimination methods to eliminate all  $\mathbf{y}$  in  $\mathcal{H}$ . This is the same as deriving a description of  $\mathcal{H}$  that does not contain the variable  $\mathbf{y}$ . Tiwary (2008) shows that deriving an explicit description of a projected polytope is NP-hard. Hence, we apply the method of Zhen and den Hertog (2015) to approximate the MVE center of  $\mathcal{H}$  without eliminating all the auxiliary variables  $\mathbf{y}$  in  $\mathcal{H}$ .

For a full-dimensional compact set  $\mathcal{H}$ , the MVE center can be obtained by solving the following optimization problem:

$$\max_{\mathbf{x}, E} \left\{ \log \det E \mid \forall \boldsymbol{\epsilon} : \|\boldsymbol{\epsilon}\|_2 \leq 1, \exists \mathbf{y}(\boldsymbol{\epsilon}) : D \begin{bmatrix} \mathbf{x} + E\boldsymbol{\epsilon} \\ \mathbf{y}(\boldsymbol{\epsilon}) \end{bmatrix} \leq \mathbf{c} \right\}, \quad (12)$$

where the vector  $\mathbf{x}$  is a non-adjustable variable and the vector  $\mathbf{y}$  is an adjustable variable. The vector function  $\mathbf{y}(\boldsymbol{\epsilon})$  is called a decision rule. Problem (12) can be considered as an adjustable robust optimization problem:

$$\begin{aligned} (AP) \quad & \max_{\mathbf{x}, \mathbf{y}(\boldsymbol{\epsilon}), E} \log \det E \\ \text{s.t.} \quad & \mathbf{d}_i^T \begin{bmatrix} \mathbf{x} + E\boldsymbol{\epsilon} \\ \mathbf{y}(\boldsymbol{\epsilon}) \end{bmatrix} \leq c_i \quad \forall \boldsymbol{\epsilon} : \|\boldsymbol{\epsilon}\|_2 \leq 1, \forall i, \end{aligned}$$

where  $\mathbf{d}_i \in \mathbb{R}^t$  is the  $i$ -th row of the matrix  $D$ ,  $E \in \mathbb{S}^n$  is a  $n \times n$  symmetric matrix and  $\boldsymbol{\epsilon} \in \mathbb{R}^n$ . Ben-Tal et al. (2004) show that in general, solving such a problem with the vector function  $\mathbf{y}(\boldsymbol{\epsilon})$

is NP-hard. We therefore restrict the vector function  $\mathbf{y}(\boldsymbol{\epsilon})$  to a given class. Despite that linear functions may not be optimal, it appears that such a decision rule performs well in practice (see Ben-Tal et al. (2009)). Let the decision rule  $\mathbf{y}(\boldsymbol{\epsilon})$  be linear, i.e.,

$$\mathbf{y} = \mathbf{u} + V\boldsymbol{\epsilon},$$

where the coefficients  $\mathbf{u} \in \mathbb{R}^s$  and  $V \in \mathbb{R}^{s \times n}$  will be optimization variables in (AAP). Let us substitute the linear decision rule of  $\mathbf{y}$  in (AP) to get the affinely adjustable robust formulation:

$$\begin{aligned} (AAP) \quad & \max_{\mathbf{x}, \mathbf{u}, E, V} \quad \log \det E \\ \text{s.t.} \quad & \mathbf{d}_i^T \begin{bmatrix} \mathbf{x} + E\boldsymbol{\epsilon} \\ \mathbf{u} + V\boldsymbol{\epsilon} \end{bmatrix} \leq c_i \quad \forall \boldsymbol{\epsilon} : \|\boldsymbol{\epsilon}\|_2 \leq 1, \forall i. \end{aligned}$$

Problem (AAP) is a semi-infinite optimization problem that approximates the MVE center. It can be reformulated into an equivalent SDP problem (see Boyd and Vandenberghe (2004)):

$$\begin{aligned} (AARP) \quad & \max_{\mathbf{x}, \mathbf{u}, E, V} \quad \log \det E \\ \text{s.t.} \quad & \mathbf{d}_i^T \begin{bmatrix} \mathbf{x} \\ \mathbf{u} \end{bmatrix} + \left\| \begin{bmatrix} E \\ V \end{bmatrix}^T \mathbf{d}_i \right\|_2 \leq c_i \quad \forall i, \end{aligned}$$

where  $\begin{bmatrix} E \\ V \end{bmatrix} \in \mathbb{R}^{t \times n}$ . We denote  $\mathbf{x}_{MVE}$  as the approximated MVE center of  $\mathcal{H}$  obtained from (AARP). In Example 3, we solve (AARP) to compute  $\mathbf{x}_{MVE}$  of the solution set  $\mathcal{H}$ .

**EXAMPLE 4. The maximum volume inscribed ellipsoid** (Example 3 continued). *Let us first eliminate  $\zeta_0$  in the set (9) by using the equality constraints, and denote the resulting full-dimensional polytope as  $\mathcal{H}$ . Then, we apply the classical method described in Boyd and Vandenberghe (2004) to find the MVE center of the set. The MVE center with respect to all the variables is at  $(\mathbf{x}^T \ \mathbf{y}^T) = (58.6, 30.7, 73.7, 20, 45.7)$ . Since we are interested in the MVE center of  $\mathcal{H}$  only with respect to  $\mathbf{x}$ , we project this MVE center onto the  $\mathbf{x}$ -space and find  $\mathbf{x}_{cm}$  at (58.6, 30.7). Lastly, we compute the  $\mathbf{x}_{MVE}$  of  $\mathcal{H}$  by solving (AARP). The resulting center  $\mathbf{x}_{MVE}$  is obtained at (52.1, 30.7). In order to evaluate the obtained solution, we derive an explicit description of  $\mathcal{H}$  with no auxiliary variables by using variable elimination methods, e.g., Fourier (1824). Note that, in general, it is NP-hard to derive such a description. The optimal MVE center  $\mathbf{x}_{opt}$  is at (53.6, 30). The  $\mathbf{x}_{MVE}$  is a much closer approximation of  $\mathbf{x}_{opt}$  than  $\mathbf{x}_{cm}$ . The robust solutions are plotted in Figure 3.*

When  $\mathcal{H}$  is unbounded, the volume of the MVE is also unbounded. The boundedness of  $\mathcal{H}$  can be checked in polynomial time. One can easily adapt the problem (AARP) to find the Chebyshev center of  $\mathcal{H}$  only with respect to  $\mathbf{x}$ . In case the polyhedral set  $\mathcal{H}$  is not full-dimensional or one wish to apply a more advanced decision rule, e.g., a quadratic decision rule, we refer the reader to Zhen and den Hertog (2015) for more details.

### 3.2. Upper Bounding Method of Hadjiyiannis et al. (2011)

This subsection is adapted from Zhen and den Hertog (2015). In Hadjiyiannis et al. (2011), authors compute upper bounds on the optimal value of adjustable robust optimization problems by only considering a finite set of scenarios, which they call *critically binding scenarios* (CBSs). The CBSs are obtained by solving the auxiliary optimization problems:

$$\boldsymbol{\epsilon}^k = \underset{\boldsymbol{\epsilon}: \|\boldsymbol{\epsilon}\|_2 \leq 1}{\operatorname{argmax}} [\mathbf{d}_{k\cdot}^1]^T \begin{bmatrix} FE^* \\ V^* \end{bmatrix} \boldsymbol{\epsilon}, \quad k = 1, \dots, l, \quad (13)$$

where  $E^*$  and  $V^*$  denote the optimal solution from (AARP). If more than one CBS is determined from the  $k$ -th constraint, an arbitrary CBS is chosen and included in the CBS set. The scenario counterpart of (AP) with respect to the CBS set  $\hat{\mathcal{U}}$ , where  $\hat{\mathcal{U}} = \{\boldsymbol{\epsilon}^1, \dots, \boldsymbol{\epsilon}^l\}$ , is given by the following optimization problem:

$$\begin{aligned} (AP - ub) \quad & \max_{\mathbf{z}, \tilde{\mathbf{y}}^k, E} \log \det E \\ \text{s.t.} \quad & [\mathbf{d}_{i\cdot}^1]^T \begin{bmatrix} F(\mathbf{z} + E\boldsymbol{\epsilon}^k) + \mathbf{x}_0 \\ \tilde{\mathbf{y}}^k \end{bmatrix} \leq b_i^1 \quad \forall i, \forall k = 1, \dots, l. \end{aligned}$$

For the  $k$ -th CBS  $\boldsymbol{\epsilon}^k \in \hat{\mathcal{U}}$ , we only need a feasible  $\tilde{\mathbf{y}}^k$  to exist. Problem (AP - ub) provides an upper bound on the optimal value of (AP), since  $\hat{\mathcal{U}} \subset \{\boldsymbol{\epsilon} : \|\boldsymbol{\epsilon}\|_2 \leq 1\}$ .

## 4. Minimize the Expected Sum of Squared Violations

In this section, we apply the approaches of Madansky (1959) and Ben-Tal and Hochman (1972) to find robust solutions for systems of uncertain linear equations.

Suppose partial information about the distributions of the uncertain parameter  $\boldsymbol{\zeta} \in \mathbb{R}^m$  is known, where the components of  $\boldsymbol{\zeta}$  are independently distributed. The partial information includes knowledge about:

- support-including intervals and means of random variables (Madansky (1959)),
- support-including intervals, means and mean absolute deviations (MADs) of random variables (Ben-Tal and Hochman (1972)).

The coefficient matrix  $A(\boldsymbol{\zeta}) : \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$  and right-hand side  $\mathbf{b}(\boldsymbol{\zeta}) : \mathbb{R}^m \rightarrow \mathbb{R}^n$  of system (4) are affine in  $\boldsymbol{\zeta}$ . The robust solution is defined as the optimal solution of problem (EWD), i.e., the minimizer of the expected sum of squared errors with respect to the worse-case distribution of  $\boldsymbol{\zeta}$ . The closed-form expression for (EP) can be derived. Then, the min-max problem (EWD) becomes a convex minimization problem only with optimization variable  $\mathbf{x}$ , which can be solved efficiently.

In §4.1 and §4.2, we briefly present the methods of Madansky (1959) and Ben-Tal and Hochman (1972). These two subsections are adapted from Postek et al. (2015). In §4.3, we discuss the application of  $\boldsymbol{\mu}$  and  $(\boldsymbol{\mu}, \mathbf{d})$  approaches to systems of uncertain linear equations.

#### 4.1. $\mu$ Approach of Madansky (1959)

Let us first consider the one-dimensional case  $\zeta \in \mathbb{R}$ . Suppose the support-including interval  $[\underline{\zeta}, \bar{\zeta}]$  and the mean  $\mu \in \mathbb{R}$  of a random variable  $\zeta$  are known. The set of probability distributions  $\mathbb{P}_\zeta$  is defined as:

$$\mathcal{P}_\mu = \{\mathbb{P}_\zeta : \text{supp}(\zeta) \subseteq [\underline{\zeta}, \bar{\zeta}], \mathbb{E}_{\mathbb{P}_\zeta}(\zeta) = \mu\}.$$

From Madansky (1959), we know that for a function  $g : \mathbb{R}^{1+n} \rightarrow \mathbb{R}$  that is convex in  $\zeta$  for all  $\mathbf{x}$ , it holds that:

$$\sup_{\mathbb{P}_\zeta \in \mathcal{P}_\mu} \mathbb{E}[g(\zeta, \mathbf{x})] = \mathbb{P}_\zeta^*(\zeta = \underline{\zeta})g(\underline{\zeta}, \mathbf{x}) + \mathbb{P}_\zeta^*(\zeta = \bar{\zeta})g(\bar{\zeta}, \mathbf{x}) \quad (14)$$

where

$$\mathbb{P}_\zeta^*(\zeta = \underline{\zeta}) = \frac{\bar{\zeta} - \mu}{\bar{\zeta} - \underline{\zeta}}, \quad \mathbb{P}_\zeta^*(\zeta = \bar{\zeta}) = \frac{\mu - \underline{\zeta}}{\bar{\zeta} - \underline{\zeta}}.$$

The worst-case probability distribution  $\mathbb{P}_\zeta^*$  in (14) is a two-point distribution with the positive mass only on the two extreme values of  $\zeta$ . The worst-case expectation of  $g(\zeta, \mathbf{x})$  is the weighted sum of some scenarios. If the functions  $g(\underline{\zeta}, \cdot)$  and  $g(\bar{\zeta}, \cdot)$  are convex, the right-hand side of (14) is convex in  $\mathbf{x}$ .

Let us now consider the multi-dimensional case. Suppose the random variable  $\boldsymbol{\zeta} \in \mathbb{R}^m$  resides in the support-including interval  $[\underline{\boldsymbol{\zeta}}, \bar{\boldsymbol{\zeta}}]$  and its mean is  $\boldsymbol{\mu} \in \mathbb{R}^m$ . The worst-case probability distribution per component of  $\boldsymbol{\zeta}$  is:

$$\mathbb{P}_{\boldsymbol{\zeta}}^*(\zeta_i = \underline{\zeta}_i) = \frac{\bar{\zeta}_i - \mu_i}{\bar{\zeta}_i - \underline{\zeta}_i}, \quad \mathbb{P}_{\boldsymbol{\zeta}}^*(\zeta_i = \bar{\zeta}_i) = \frac{\mu_i - \underline{\zeta}_i}{\bar{\zeta}_i - \underline{\zeta}_i} \quad \forall i = 1, \dots, m.$$

The worst-case distribution  $\mathbb{P}_{\boldsymbol{\zeta}}^*$  only involves the two extreme values of  $\zeta_i \in [\underline{\zeta}_i, \bar{\zeta}_i]$ , i.e.,  $\underline{\zeta}_i$  and  $\bar{\zeta}_i$ , for  $i = 1, \dots, m$ . There are  $2^m$  possible combinations of the extreme values of  $\boldsymbol{\zeta}$ , denoted as  $\boldsymbol{\zeta}^{(i)}, i = 1, \dots, 2^m$ . Let  $\mathcal{S}$  denote the set of the possible combinations:

$$\mathcal{S} = \{\boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}, \dots, \boldsymbol{\zeta}^{(2^m)}\}.$$

For any function  $g : \mathbb{R}^{m+n} \rightarrow \mathbb{R}$  that is convex in  $\boldsymbol{\zeta}$  for all  $\mathbf{x}$ , it holds that:

$$\sup_{\mathbb{P}_{\boldsymbol{\zeta}} \in \mathcal{P}_{\boldsymbol{\mu}}} \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}} [g(\boldsymbol{\zeta}, \mathbf{x})] = \sum_{i=1}^{2^m} \mathbb{P}_{\boldsymbol{\zeta}}^*(\boldsymbol{\zeta} = \boldsymbol{\zeta}^{(i)})g(\boldsymbol{\zeta}^{(i)}, \mathbf{x}), \quad (15)$$

where  $\mathbb{P}_{\boldsymbol{\zeta}}^*(\boldsymbol{\zeta} = \boldsymbol{\zeta}^{(i)}) = \prod_{j=1}^m \mathbb{P}_{\zeta_j}^*(\zeta_j = \zeta_j^{(i)})$ , due to the independency among the components of  $\boldsymbol{\zeta}$ . Again, if the functions  $g(\boldsymbol{\zeta}^{(i)}, \cdot)$  are convex, for all  $i$ , the right-hand side of (15) is convex in  $\mathbf{x}$ . This function is a sum of  $2^m$  convex functions.

#### 4.2. $(\mu, d)$ Approach of Ben-Tal and Hochman (1972)

Let us first consider the one-dimensional case. Suppose the support-including interval  $[\underline{\zeta}, \bar{\zeta}]$ , the mean  $\mu \in \mathbb{R}$  and the MAD  $d \in \mathbb{R}$  of a random variable  $\zeta \in \mathbb{R}$  is known. The set of probability distributions  $\mathbb{P}_\zeta$  is defined as:

$$\mathcal{P}_{\mu,d} = \{\mathbb{P}_\zeta : \text{supp}(\zeta) \subseteq [\underline{\zeta}, \bar{\zeta}], \mathbb{E}_{\mathbb{P}_\zeta}(\zeta) = \mu, \mathbb{E}_{\mathbb{P}_\zeta}(|\zeta - \mu|) = d\}. \quad (16)$$

From Ben-Tal and Hochman (1972), we know that for a function  $g: \mathbb{R}^{1+n} \rightarrow \mathbb{R}$  that is convex in  $\zeta$  for all  $\mathbf{x}$ , it holds that:

$$\sup_{\mathbb{P}_\zeta \in \mathcal{P}_{\mu,d}} \mathbb{E}_{\mathbb{P}_\zeta} [g(\zeta, \mathbf{x})] = \mathbb{P}_\zeta^*(\zeta = \underline{\zeta})g(\underline{\zeta}, \mathbf{x}) + \mathbb{P}_\zeta^*(\zeta = \bar{\zeta})g(\bar{\zeta}, \mathbf{x}) + \mathbb{P}_\zeta^*(\zeta = \mu)g(\mu, \mathbf{x}) \quad (17)$$

where

$$\mathbb{P}_\zeta^*(\zeta = \underline{\zeta}) = \frac{d}{2(\mu - \underline{\zeta})}, \quad \mathbb{P}_\zeta^*(\zeta = \bar{\zeta}) = \frac{d}{2(\bar{\zeta} - \mu)}, \quad \mathbb{P}_\zeta^*(\zeta = \mu) = 1 - \frac{d}{2(\mu - \underline{\zeta})} - \frac{d}{2(\bar{\zeta} - \mu)}.$$

The worst-case distribution  $\mathbb{P}_\zeta^*$  is a three-point distribution with positive probability only on the two extreme values and the mean of  $\zeta$ . The worst-case expectation of  $g(\zeta, \mathbf{x})$  is the sum of weighted scenarios. If the functions  $g(\underline{\zeta}, \cdot)$ ,  $g(\bar{\zeta}, \cdot)$  and  $g(\mu, \cdot)$  are convex, the right-hand side of (17) is convex in  $\mathbf{x}$ .

Let us now consider the multi-dimensional case. Suppose the random variable  $\boldsymbol{\zeta} \in \mathbb{R}^m$  resides in the support-including interval  $[\underline{\boldsymbol{\zeta}}, \bar{\boldsymbol{\zeta}}]$ , and its mean and MAD are  $\boldsymbol{\mu} \in \mathbb{R}^m$  and  $\mathbf{d} \in \mathbb{R}^m$ , respectively. For  $i = 1, \dots, m$ , the worst-case probability distribution of  $\zeta_i$  is:

$$\begin{aligned} \mathbb{P}_{\boldsymbol{\zeta}}^*(\zeta_i = \underline{\zeta}_i) &= \frac{d_i}{2(\mu_i - \underline{\zeta}_i)}, \quad \mathbb{P}_{\boldsymbol{\zeta}}^*(\zeta_i = \bar{\zeta}_i) = \frac{d_i}{2(\bar{\zeta}_i - \mu_i)}, \\ \mathbb{P}_{\boldsymbol{\zeta}}^*(\zeta_i = \mu_i) &= 1 - \frac{d_i}{2(\mu_i - \underline{\zeta}_i)} - \frac{d_i}{2(\bar{\zeta}_i - \mu_i)}. \end{aligned}$$

The worst-case distribution  $\mathbb{P}_{\boldsymbol{\zeta}}^*$  only involves three possible values of  $\zeta_i \in [\underline{\zeta}_i, \bar{\zeta}_i]$ , i.e.,  $\underline{\zeta}_i$ ,  $\mu_i$  and  $\bar{\zeta}_i$ , for  $i = 1, \dots, m$ . There are  $3^m$  possible combinations of the three values of  $\boldsymbol{\zeta}$ , denoted as  $\boldsymbol{\zeta}^{(i)}$ ,  $i = 1, \dots, 3^m$ .

Let  $\mathcal{S}$  denote the following set:

$$\mathcal{S} = \{\boldsymbol{\zeta}^{(1)}, \boldsymbol{\zeta}^{(2)}, \dots, \boldsymbol{\zeta}^{(3^m)}\}.$$

For any function  $g: \mathbb{R}^{m+n} \rightarrow \mathbb{R}$  that is convex in  $\boldsymbol{\zeta}$  for all  $\mathbf{x}$ , it holds that:

$$\sup_{\mathbb{P}_{\boldsymbol{\zeta}} \in \mathcal{P}_{\boldsymbol{\mu}, \mathbf{d}}} \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}} [g(\boldsymbol{\zeta}, \mathbf{x})] = \sum_{i=1}^{3^m} \mathbb{P}_{\boldsymbol{\zeta}}^*(\boldsymbol{\zeta} = \boldsymbol{\zeta}^{(i)})g(\boldsymbol{\zeta}^{(i)}, \mathbf{x}), \quad (18)$$

where  $\mathbb{P}_{\boldsymbol{\zeta}}^*(\boldsymbol{\zeta} = \boldsymbol{\zeta}^{(i)}) = \prod_{j=1}^m \mathbb{P}_{\boldsymbol{\zeta}}^*(\zeta_j = \zeta_j^{(i)})$  due to the independency among the components of  $\boldsymbol{\zeta}$ . Again, if the functions  $g(\boldsymbol{\zeta}^{(i)}, \cdot)$  are convex, for all  $i$ , the right-hand side of (18) is convex in  $\mathbf{x}$ . This function is a sum of  $3^m$  convex functions.

### 4.3. $\mu$ and $(\mu, d)$ Approaches for Systems of Uncertain Linear Equations

In general, the closed-form expression of  $\mu$  and  $(\mu, d)$  approaches in (15) and (18) involves a sum which the size grows exponentially in the dimension  $m$ . For problems with a high value for  $m$ , the  $\mu$  and  $(\mu, d)$  approaches may be computationally intractable. In this subsection, we apply the  $(\mu, d)$  approach to determine the minimizer  $\mathbf{x}_{(\mu, d)}$  of the problem  $(EWD)$  (see §1). We obtain that the size of the closed-form expression for  $(EP)$  increases quadratically in  $m$ .

We assume that the support-including interval, mean and MAD of the random variable  $\zeta \in \mathbb{R}^m$  are known. Moreover, the components of  $\zeta$  are independently distributed, and the components of  $A: \mathbb{R}^m \rightarrow \mathbb{R}^{n \times n}$  and  $\mathbf{b}: \mathbb{R}^m \rightarrow \mathbb{R}^n$  are affine in  $\zeta \in \mathbb{R}^m$ . Note that we do not impose any structures (e.g., element-wise or column-wise) on the uncertainties in  $A$  and  $\mathbf{b}$ . The function:

$$g(\zeta, \mathbf{x}) = \|A(\zeta)\mathbf{x} - \mathbf{b}(\zeta)\|_2^2$$

is clearly convex in  $\zeta$  for all  $\mathbf{x}$ . Problem  $(EP)$  (see §1) with respect to the worst-case distribution  $\mathbb{P}_\zeta^*$  becomes:

$$\mathbb{E}_{\mathbb{P}_\zeta^*} [\|A(\zeta)\mathbf{x} - \mathbf{b}(\zeta)\|_2^2]. \quad (19)$$

Expression (19) is the expected sum of squared violations with probability distribution  $\mathbb{P}_\zeta^*$ . It can be rewritten as an expected sum of at most  $m(m+1)/2$  quadratic terms and  $m$  linear terms in  $\zeta$ . From §4.2, we know that the expectation of each quadratic term can be expressed as a sum of at most nine terms. Similarly, the expectation of each linear term can be expressed as one term. Together with a constant term, we can express the expectation in (19) as a sum of at most  $m(9m-1)/2 + 1$  terms. This significantly reduces the complexity of the  $(\mu, d)$  approach from  $\mathcal{O}(3^m)$  to  $\mathcal{O}(m^2)$  and allows us to solve  $(EWD)$  in polynomial time. A similar reduction holds for the  $\mu$  approach. In Example 5, we apply the  $\mu$  and  $(\mu, d)$  approaches to find robust solutions for an interval linear system with known support-including intervals, means and/or MADs of the uncertain parameters.

**EXAMPLE 5. The robust solutions for a interval linear system.** *Let us consider the following distributional version of the interval linear system in Example 3:*

$$\begin{bmatrix} \zeta_1 & \zeta_2 \\ 2 & \zeta_3 \end{bmatrix} \mathbf{x} = \begin{bmatrix} \zeta_4 \\ \zeta_5 \end{bmatrix},$$

where the support-including interval, mean and MAD of the uncertain parameter  $\zeta = [\zeta_1 \ \zeta_2 \ \dots \ \zeta_5]^T \in \mathbb{R}^5$  are known, and the set of probability distributions of  $\zeta$  is defined as follows:

$$\mathcal{P}_{\mu, d} = \left\{ \mathbb{P}_\zeta : \begin{array}{lll} \text{supp}(\zeta_1) \subseteq [0, 1], & \mathbb{E}_{\mathbb{P}_\zeta}[\zeta_1] = \frac{1}{2}, & \mathbb{E}_{\mathbb{P}_\zeta} \left[ \left| \zeta_1 - \frac{1}{2} \right| \right] = \frac{3}{10} \\ \text{supp}(\zeta_2) \subseteq [2, 3], & \mathbb{E}_{\mathbb{P}_\zeta}[\zeta_2] = \frac{5}{2}, & \mathbb{E}_{\mathbb{P}_\zeta} \left[ \left| \zeta_2 - \frac{5}{2} \right| \right] = \frac{3}{10} \\ \text{supp}(\zeta_3) \subseteq [1, 2], & \mathbb{E}_{\mathbb{P}_\zeta}[\zeta_3] = \frac{3}{2}, & \mathbb{E}_{\mathbb{P}_\zeta} \left[ \left| \zeta_3 - \frac{3}{2} \right| \right] = \frac{3}{10} \\ \text{supp}(\zeta_4) \subseteq [0, 120], & \mathbb{E}_{\mathbb{P}_\zeta}[\zeta_4] = 60, & \mathbb{E}_{\mathbb{P}_\zeta} [|\zeta_4 - 60|] = 36 \\ \text{supp}(\zeta_5) \subseteq [60, 240], & \mathbb{E}_{\mathbb{P}_\zeta}[\zeta_5] = 150, & \mathbb{E}_{\mathbb{P}_\zeta} [|\zeta_5 - 150|] = 54 \end{array} \right\}.$$



We apply the  $(\boldsymbol{\mu}, \mathbf{d})$  approach and solve problem (EWD) to find a robust solution for this distributional interval linear system. The corresponding (EP) is as follows:

$$\sup_{\mathbb{P}_{\boldsymbol{\zeta}} \in \mathcal{P}_{\boldsymbol{\mu}, \mathbf{d}}} \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}} \left[ \left\| \begin{bmatrix} \zeta_1 & \zeta_2 \\ 2 & \zeta_3 \end{bmatrix} \mathbf{x} - \begin{bmatrix} \zeta_4 \\ \zeta_5 \end{bmatrix} \right\|_2^2 \right].$$

Note that if we apply the  $(\boldsymbol{\mu}, \mathbf{d})$  approach directly, the closed-form expression of (EP) is a sum of 243 (i.e.,  $3^5$ ) quadratic functions. We first determine the worst-case distribution  $\mathbb{P}_{\boldsymbol{\zeta}}^* \in \mathcal{P}_{\boldsymbol{\mu}, \mathbf{d}}$ , and express (EP) as:

$$\mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [(\zeta_1 x_1 + \zeta_2 x_2 - \zeta_4)^2] + \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [(2x_1 + \zeta_3 x_2 - \zeta_5)^2],$$

where

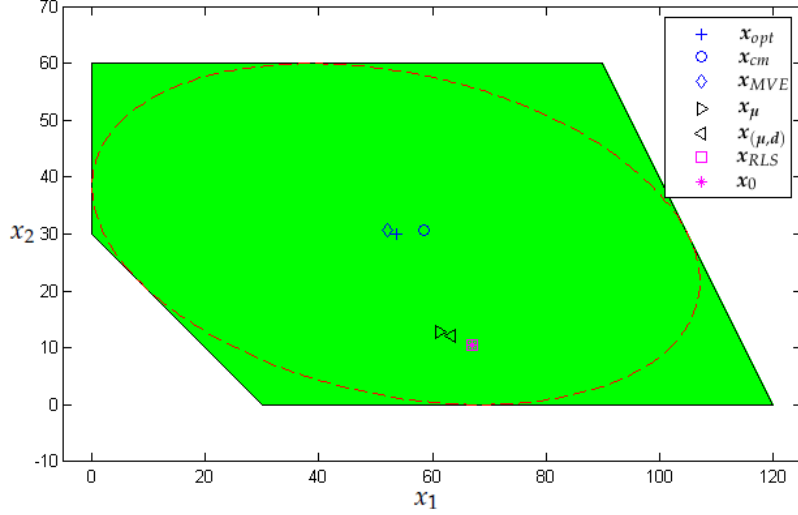
$$\begin{aligned} \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [(\zeta_1 x_1 + \zeta_2 x_2 - \zeta_4)^2] &= \underbrace{\mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [\zeta_1^2 x_1^2] + \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [\zeta_2^2 x_2^2] + \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [\zeta_4^2]}_{3 \text{ terms}} + \underbrace{\mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [2\zeta_1 \zeta_2 x_1 x_2]}_{9 \text{ terms}} \\ &\quad - \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [2\zeta_1 \zeta_4 x_1] - \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [2\zeta_2 \zeta_4 x_2], \\ \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [(2x_1 + \zeta_3 x_2 - \zeta_5)^2] &= 4x_1^2 + \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [\zeta_3^2 x_2^2] + \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [\zeta_5^2] + \underbrace{\mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [4\zeta_3 x_1 x_2]}_{1 \text{ term}} \\ &\quad - \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [4\zeta_5 x_1] - \mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [2\zeta_3 \zeta_5 x_2]. \end{aligned}$$

From the  $(\boldsymbol{\mu}, \mathbf{d})$  approach, we know that, for instance, the expectation term that contains  $\zeta_i \zeta_j$ ,  $i \neq j$ , can be expressed as a sum of 9 terms. In total, the simplified closed-form expression for (EP) is a sum of 54 ( $\ll 243$ ) terms. By minimizing the obtained sum over  $\mathbf{x}$ , we find the robust solution  $\mathbf{x}_{\boldsymbol{\mu}, \mathbf{d}}$  at (63.4, 12). Similarly, the robust solution from the  $\boldsymbol{\mu}$  approach is obtained at  $\mathbf{x}_{\boldsymbol{\mu}} = (61.3, 12.7)$ . In Figure 1, the robust solutions from the  $\boldsymbol{\mu}$  and  $(\boldsymbol{\mu}, \mathbf{d})$  approaches are denoted as  $\mathbf{x}_{\boldsymbol{\mu}}$  and  $\mathbf{x}_{(\boldsymbol{\mu}, \mathbf{d})}$ , respectively.

In general, after the worst-case distribution is determined, one can reduce the complexity of the  $\boldsymbol{\mu}$  and  $(\boldsymbol{\mu}, \mathbf{d})$  approaches by considering, if possible, the sum of the expectation of each term instead of the expectation of the sum altogether (see Postek et al. (2015)). However, this is not possible for, e.g.,  $\mathbb{E}_{\mathbb{P}_{\boldsymbol{\zeta}}^*} [\|A(\boldsymbol{\zeta})\mathbf{x} - \mathbf{b}(\boldsymbol{\zeta})\|_2]$ .

## 5. Comparison of Robust Solution Methods

In this section, we compare the theoretical aspects of the RLS method with two new robust solution methods discussed in §3 and §4. The comparative advantages and disadvantages of the three robust solution methods are also summarized in Table 10 (see Appendix). First, we determine the RLS solution for the interval linear system in Example 3.

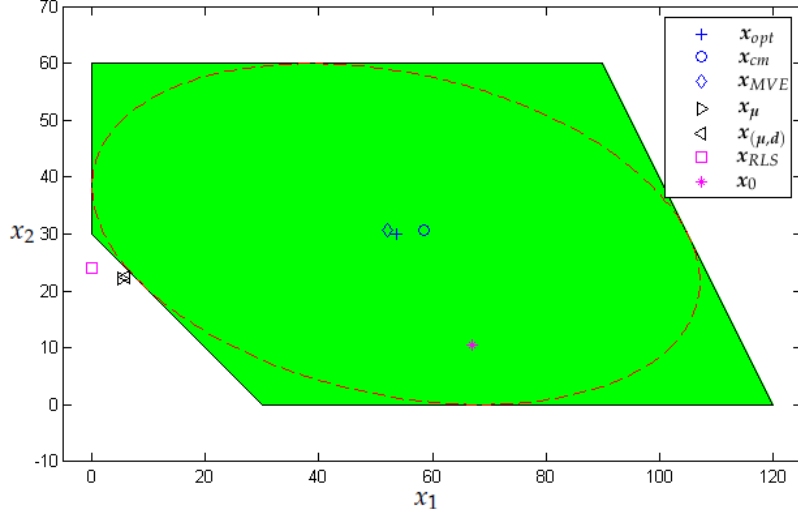


**Figure 1** The shaded region is the solution set  $\mathcal{X}$  in Example 3. The dashed ellipsoid is the maximum volume inscribed ellipsoid. The solutions from the MVE method,  $\mu$  approach,  $(\mu, d)$  approach and RLS method are denoted as  $\mathbf{x}_{MVE}$ ,  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu,d)}$  and  $\mathbf{x}_{RLS}$ , respectively. The solutions  $\mathbf{x}_0$  and  $\mathbf{x}_{opt}$  are the nominal solution and the optimal MVE center, respectively. The solution  $\mathbf{x}_{cm}$  is the ellipsoid center from the classical method.

**EXAMPLE 6. The robust least-squares solution for an interval linear system.** We apply RLS method to the interval linear system in Example 3 and find the robust solution  $\mathbf{x}_{RLS}$  is at  $(67.06, 10.59)$ , which is denoted as “ $\square$ ” in Figure 1. The solution  $\mathbf{x}_{RLS}$  coincides with the nominal solution  $\mathbf{x}_0$  of the system.

The RLS method,  $\mu$  and  $(\mu, d)$  approaches are in line with the philosophy of Robust Optimization (see Ben-Tal et al. (2009)), i.e., minimizing the violation with respect to the worst-case scenario. The difference is that the RLS method considers the worst-case  $\zeta$ , whereas the  $\mu$  and  $(\mu, d)$  approaches consider the expected value with respect to the worst-case distribution of  $\zeta$ . The solution  $\mathbf{x}_{MVE}$  from the MVE method is a centralized solution of the solution set.

The uncertainty sets of the robust solution methods require specific structures to remain computationally tractable. The robust solution  $\mathbf{x}_{MVE}$  can be obtained efficiently if the column-wise uncertainties in  $A(\zeta)$  and  $\mathbf{b}(\zeta)$  reside in polyhedral uncertainty sets, i.e., the uncertainties within each column of  $A(\zeta)$  and  $\mathbf{b}(\zeta)$  may be affinely dependent. In many real-life problems that involve solving a system of linear equations, the uncertainties are often column-wise (see §6). In general, the closed-form expression of the  $\mu$  and  $(\mu, d)$  approaches involve a sum which the size grows exponentially in the dimension  $m$ . However, problem (EWD) with affine uncertainties in  $A(\zeta)$  and  $\mathbf{b}(\zeta)$  can be solved efficiently. Note that besides the restriction that the components of  $\zeta$  are independent, no further restrictions are imposed on the structure of the uncertainties in  $A(\zeta)$  and



**Figure 2** The shaded region is the solution set  $\mathcal{X}$  in Example 3. The dashed ellipsoid is the maximum volume inscribed ellipsoid. The solutions from the MVE method,  $\mu$  approach,  $(\mu, d)$  approach and RLS method are denoted as  $\mathbf{x}_{MVE}$ ,  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu,d)}$  and  $\mathbf{x}_{RLS}$ , respectively. The solutions  $\mathbf{x}_0$  and  $\mathbf{x}_{opt}$  are the nominal solution and the optimal MVE center, respectively. The solution  $\mathbf{x}_{cm}$  is the ellipsoid center from the classical method.

$\mathbf{b}(\zeta)$ , even not column-wise uncertainty. However, the  $\mu$  and  $(\mu, d)$  approaches can only deal with limited statistical information about  $\zeta$ . This restricts the modeling power of the  $\mu$  and  $(\mu, d)$  approaches. In case of the RLS method, the optimal solution can be obtained by solving an SOCP or an SDP problem if the uncertainty sets follow some special structures. The limited choices of the uncertainty sets limit the flexibility of the RLS method.

One of the most fundamental properties of a system of (uncertain) linear equations is scale invariance. The nominal solution  $\mathbf{x}_0$  and MVE solution  $\mathbf{x}_{MVE}$  are scale invariant. The MVE solutions are in the (relative) interior of the solution set. Hence, there exists a  $\zeta \in \mathcal{U}$ , such that  $A(\zeta)\mathbf{x}_{MVE} = \mathbf{b}(\zeta)$ . However, the robust solutions  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu,d)}$  and  $\mathbf{x}_{RLS}$  are not scale invariant. Moreover, as it is shown in Example 7, these solutions can be outside the solution set. As the feasibility of the solutions are not guaranteed,  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu,d)}$  and  $\mathbf{x}_{RLS}$  are theoretically less appealing than  $\mathbf{x}_{MVE}$ .

**EXAMPLE 7. Scale sensitivity of the robust solutions.** *Let us consider an adapted version of the interval linear system in Example 3 where the uncertainty sets are defined as:*

$$\mathcal{U}_0 = \left\{ \zeta_0 : \begin{bmatrix} 0 \\ 60 \end{bmatrix} \leq \zeta_0 \leq \begin{bmatrix} 3600 \\ 240 \end{bmatrix} \right\}, \mathcal{U}_1 = \left\{ \zeta_1 : \begin{bmatrix} 0 \\ 2 \end{bmatrix} \leq \zeta_1 \leq \begin{bmatrix} 30 \\ 2 \end{bmatrix} \right\}, \mathcal{U}_2 = \left\{ \zeta_2 : \begin{bmatrix} 60 \\ 1 \end{bmatrix} \leq \zeta_2 \leq \begin{bmatrix} 90 \\ 2 \end{bmatrix} \right\}.$$

*The components of the first row of the interval linear system in Example 3 are now multiplied by a factor 30. Note that this operation does not alter the set  $\mathcal{X} \cap \mathbb{R}_+^n$ . We accordingly adjust the corresponding support-including intervals, means and MADs of  $\zeta_1, \zeta_2$  and  $\zeta_4$  in Example 5. For example,*

**Table 1** The applicability of the robust solution methods with respect to input-output model, Colley’s Matrix Ranking and Article Influence Scores. The “✓” means the robust solution method is applicable; the “✗” means inapplicable; the “–” means the robust solution method is applicable but not computationally tractable.

	$\mathbf{x}_{MVE}$	$\mathbf{x}_\mu$	$\mathbf{x}_{(\mu,d)}$	$\mathbf{x}_{RLS}$
Input-output Model	✓	✓	✓	✓
Colley’s Matrix Ranking	✓	✗	✗	–
AIS Ranking	✓	✗	✗	✗

the support-including interval, mean and MAD of  $\zeta_4$  become  $\text{supp}(\zeta_4) \subseteq [0, 3600]$ ,  $\mathbb{E}_{\mathbb{P}_\zeta}(\zeta_4) = 1800$  and  $\mathbb{E}_{\mathbb{P}_\zeta}(|\zeta_4 - 1800|) = 1080$ , respectively. The robust solutions for this uncertainty set is depicted in Figure 2. The MVE solution remains unchanged. The solution  $\mathbf{x}_{RLS}$  no longer coincides with the nominal solution. The solutions  $\mathbf{x}_{RLS}$ ,  $\mathbf{x}_\mu$  and  $\mathbf{x}_{(\mu,d)}$  are outside the solution set.

## 6. Numerical Experiments

In this section, we conduct four experiments to evaluate the robustness of the robust solutions. The first experiment considers the interval linear system introduced in Example 3 and its adapted version in Example 7. The other three are input-output model, Colley’s Matrix Ranking and Article Influence Scores, respectively. A common feature of these problems is that their solutions are obtained by solving a system of linear equations. Here, we assume the systems are uncertain:

- Input-output Model: independent interval uncertainties in  $A(\zeta)$  and  $\mathbf{b}(\zeta)$
- Colley’s Matrix Ranking: column-wise (dependent) uncertainties in  $\mathbf{b}(\zeta)$
- Article Influence Scores: column-wise (dependent) uncertainties in  $A(\zeta)$ .

In Table 1, we present the applicability of the robust solution methods with respect to input-output model, Colley’s Matrix Ranking and Article Influence Scores.

### 6.1. A Simple Experiment

Firstly, we apply the described robust solution methods to find robust solutions for the interval linear system introduced in Example 3. For a solution  $\tilde{\mathbf{x}}$ , we consider five robustness measures:

- *Volume*: the approximated volume of the MVE centered at  $\tilde{\mathbf{x}}$  within  $\mathcal{X}$
- *MDW $_\mu$* : the estimated mean of the sum of squared deviations of  $A(\zeta)\tilde{\mathbf{x}}$  from  $\mathbf{b}(\zeta)$  with respect to  $10^4$  sampled  $\zeta$  from the worst-case distributions  $\mathbb{P}_\zeta^* \in \mathcal{P}_\mu$  (i.e.,  $\mathbb{E}_{\mathbb{P}_\zeta^*} \|A(\zeta)\tilde{\mathbf{x}} - \mathbf{b}(\zeta)\|_2$ )
- *MDW $_{(\mu,d)}$* : the estimated mean of the sum of squared deviations of  $A(\zeta)\tilde{\mathbf{x}}$  from  $\mathbf{b}(\zeta)$  with respect to  $10^4$  sampled  $\zeta$  from the worst-case distribution  $\mathbb{P}_\zeta^* \in \mathcal{P}_{(\mu,d)}$  (i.e.,  $\mathbb{E}_{\mathbb{P}_\zeta^*} \|A(\zeta)\tilde{\mathbf{x}} - \mathbf{b}(\zeta)\|_2$ )
- *WCD*: the worst-case 2-norm deviations of  $A(\zeta)\tilde{\mathbf{x}}$  from  $\mathbf{b}(\zeta)$  (i.e.,  $\max_{\zeta \in \mathcal{U}} \|A(\zeta)\tilde{\mathbf{x}} - \mathbf{b}(\zeta)\|_2$ )
- *MDFS*: the estimated mean 2-norm deviations of uniformly distributed feasible solutions in the solution set (i.e.,  $\mathbb{E}_{\mathbb{P}_{\mathbf{x}}^{uni}} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2$ , where  $\mathbb{P}_{\mathbf{x}}^{uni}$  denotes the uniform distribution) by using the *Hit-and-Run* algorithm (see Smith (1984)).

**Table 2** Numerical comparison of the robust solutions for the interval linear system in Example 3. The solutions from the MVE method,  $\mu$  approach,  $(\mu, d)$  approach and RLS method are denoted as  $\mathbf{x}_{MVE}$ ,  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu, d)}$  and  $\mathbf{x}_{RLS}$ , respectively. The solutions  $\mathbf{x}_0$  and  $\mathbf{x}_{opt}$  are the nominal solution and the optimal MVE center, respectively. The bold numbers show that the corresponding robust solution performs the best (among the nominal and robust solutions) with respect to the corresponding robustness measure.

	$\mathbf{x}_0$	$\mathbf{x}_{opt}$	$\mathbf{x}_{MVE}$	$\mathbf{x}_\mu$	$\mathbf{x}_{(\mu, d)}$	$\mathbf{x}_{RLS}$	$[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$
$x_1$	67.1	53.6	52.1	61.3	63.4	67.1	$[0, 120]$
$x_2$	10.6	30	30.7	12.7	12	10.6	$[0, 60]$
<i>Volume</i>	22.444	39.279	<b>38.396</b>	23.651	23.325	22.444	–
$MDW_\mu/10^3$	12.880	14.618	14.684	<b>12.796</b>	12.807	12.880	–
$MDW_{(\mu, d)}/10^3$	7.728	9.471	9.544	7.707	<b>7.696</b>	7.728	–
$WCD/10$	<b>13.728</b>	17.914	17.874	14.434	14.181	<b>13.728</b>	–
<i>MDFS</i>	37.217	31.295	<b>31.358</b>	35.424	35.977	37.217	–
Complexity	Easy	SDP	SDP	QP	QP	SOCP	LP

In Table 2, the nominal and robust solutions and their robustness measure values are reported. The nominal solution  $\mathbf{x}_0$  is the same as  $\mathbf{x}_{RLS}$ . The exact ranges of the components of the solution  $\mathbf{x}$  can be obtained by solving some LPs. The solutions  $\mathbf{x}_0$  and  $\mathbf{x}_{RLS}$  are the best robust solutions with respect to the measure  $WCD$ . They are very robust against the worst-case deviations. Geometrically, however, they are not robust as they have the lowest *Volume* and highest *MDFS*. The robust solution  $\mathbf{x}_{(\mu, d)}$  has the best (i.e., least)  $MDW_{(\mu, d)}$  and the second best  $MDW_\mu$ . Contrastingly,  $\mathbf{x}_\mu$  has the best  $MDW_\mu$  and the second best  $MDW_{(\mu, d)}$ . The robust solution  $\mathbf{x}_{opt}$  and  $\mathbf{x}_{MVE}$  are centralized solutions. The solution  $\mathbf{x}_{opt}$  has the largest ellipsoid and the best (i.e., least) *MDFS*. The small difference in the *Volumes* of  $\mathbf{x}_{opt}$  and  $\mathbf{x}_{MVE}$  indicates that the solution  $\mathbf{x}_{MVE}$  is a very close approximation of  $\mathbf{x}_{opt}$ . Both  $\mathbf{x}_{opt}$  and  $\mathbf{x}_{MVE}$  perform poorly with respect to  $MDW_\mu$ ,  $MDW_{(\mu, d)}$  and  $WCD$ .

**Table 3** Numerical comparison of the robust solutions for the interval linear system in Example 7. The solutions from the MVE method,  $\mu$  approach,  $(\mu, d)$  approach and RLS method are denoted as  $\mathbf{x}_{MVE}$ ,  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu, d)}$  and  $\mathbf{x}_{RLS}$ , respectively. The solutions  $\mathbf{x}_0$  and  $\mathbf{x}_{opt}$  are the nominal solution and the optimal MVE center, respectively. The bold numbers show that the corresponding robust solution performs the best (among the nominal and robust solutions) with respect to the corresponding robustness measure.

	$\mathbf{x}_0$	$\mathbf{x}_{opt}$	$\mathbf{x}_{MVE}$	$\mathbf{x}_\mu$	$\mathbf{x}_{(\mu, d)}$	$\mathbf{x}_{RLS}$	$[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$
$x_1$	67.1	53.6	52.1	5.2	5.8	0	$[0, 120]$
$x_2$	10.6	30	30.7	22.1	22.3	24	$[0, 60]$
<i>Volume</i>	22.444	39.279	<b>38.396</b>	0	0	0	–
$MDW_\mu/10^5$	42.852	56.680	57.213	<b>33.797</b>	33.805	33.908	–
$MDW_{(\mu, d)}/10^5$	25.711	40.294	40.929	20.340	<b>20.333</b>	20.397	–
$WCD/10^2$	29.662	43.085	43.279	22.836	22.695	<b>21.708</b>	–
<i>MDFS</i>	37.217	31.295	<b>31.358</b>	55.162	54.632	59.315	–
Complexity	Easy	SDP	SDP	QP	QP	SOCP	LP

In Table 3, we present the numerical results for the interval linear system in Example 7. Here, the nominal solution  $\mathbf{x}_0$  is no longer the same as  $\mathbf{x}_{RLS}$ . The solutions  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu,d)}$  and  $\mathbf{x}_{RLS}$  are outside the solution set. Therefore, their corresponding volumes of the MVE are 0. The solutions  $\mathbf{x}_0$ ,  $\mathbf{x}_{opt}$  and  $\mathbf{x}_{MVE}$  are scale invariant.

## 6.2. Robust Production Vector for Input-output Model

Leontief's Nobel prize-winning input-output model describes a simplified view of an economy. Its goal is to predict the proper level of production for each of several types of goods or service. We apply this to predict the production of different industries in the Netherlands. In Table 4, we present the data that are reported in Deloitte (2014). This is a simplified version of the consumption data of the Netherlands published by the Dutch statistics office. From Leontief (1986), the nominal

**Table 4** The simplified consumption matrix of the Netherlands (numbers in €m). Five industries are considered, i.e., agriculture, fishing, forestry (AFF) industry, manufacturing industry, service industry, education and healthcare (E & H) industry and other industries. The external demand (ED) and the total output are also reported.

$C$	AFF	Manuf.	Services	E & H	Other	ED ( $\mathbf{b}_0$ )
AFF	4.257	9.828	0.221	0.092	0.476	13.232
Manuf.	8.074	114.955	14.864	4.61	33.212	296.826
Services	1.983	29.3	65.9	5.925	42.493	176.933
E & H	0.019	1.035	0.982	2.281	1.755	92.926
Other	0.628	9.425	14.871	5.431	28.366	214.992
Total output ( $\mathbf{w}^T$ )	28.193	461.369	273.771	111.265	321.294	

input-output matrix is defined as:

$$A_0 = \text{Diag}(\mathbf{w})^{-1}C,$$

where  $\mathbf{w} \in \mathbb{R}^5$  is the total output vector,  $C \in \mathbb{R}^{5 \times 5}$  is the consumption matrix from Table 4, and  $\text{Diag}(\cdot)$  places its vector components into a diagonal matrix. The nominal production vector  $\mathbf{x}_0$  can be obtained by solving the following system of linear equations:

$$(I - A_0)\mathbf{x}_0 = \mathbf{b}_0, \quad (20)$$

where  $\mathbf{b}_0$  is the vector of the nominal external demands (see the last column of Table 4).

Suppose there are uncertainties in the system (20), and each component of  $A(\boldsymbol{\zeta}) = [\zeta_1 \cdots \zeta_n]$  and  $\mathbf{b}(\boldsymbol{\zeta}) = \boldsymbol{\zeta}_0$  resides in an independent interval, where  $\boldsymbol{\zeta} = [\zeta_0^T \zeta_1^T \cdots \zeta_n^T]^T \in \mathbb{R}^{n^2+n}$ . We assume that the corresponding interval uncertainty set  $\mathcal{U}_j$ , for  $j = 1, \dots, n$ , and the distribution sets  $\mathbb{P}_\mu$  and  $\mathbb{P}_{(\mu,d)}$  are as follows:

$$\mathcal{U}_j = \{ \zeta_j : |\zeta_j - (\mathbf{a}_0)_{\cdot j}| \leq \sigma(\mathbf{a}_0)_{\cdot j} \} \quad \text{and} \quad \mathcal{U}_0 = \{ \zeta_0 : |\zeta_0 - \mathbf{b}_0| \leq \sigma \mathbf{b}_0 \}$$

$$\mathcal{P}_\mu = \left\{ \mathbb{P}_\zeta : \begin{array}{l} \mathbb{E}_\mathbb{P}(\zeta_j) = (\mathbf{a}_0)_{\cdot j}, \mathbb{E}_\mathbb{P}(\zeta_0) = \mathbf{b}_0 \\ \text{supp}(\zeta_0) \in [(1-\sigma)\mathbf{b}_0, (1+\sigma)\mathbf{b}_0], \quad \forall j \\ \text{supp}(\zeta_j) \in [(1-\sigma)(\mathbf{a}_0)_{\cdot j}, (1+\sigma)(\mathbf{a}_0)_{\cdot j}], \end{array} \right\}$$

$$\mathcal{P}_{\mu,d} = \{ \mathbb{P}_\zeta : \mathbb{E}_\mathbb{P}(|\zeta_j - (\mathbf{a}_0)_{\cdot j}|) = \sigma\gamma(\mathbf{a}_0)_{\cdot j}, \mathbb{E}_\mathbb{P}(|\zeta_0 - \mathbf{b}_0|) = \sigma\gamma\mathbf{b}_0 \} \cap \mathcal{P}_\mu,$$

where  $\sigma$  and  $\gamma$  are user specified,  $(\mathbf{a}_0)_{\cdot j}$  is the  $j$ -th column of the nominal matrix  $A_0$ . The three robust methods can be applied to this interval linear system. In Table 5, the nominal, robust production vectors and the exact ranges of the components of the production vectors  $\mathbf{x}$  are reported. The width of  $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$  indicates the sensitivities of the system with respect to the assumed uncertainties. All the procedures are performed by using SDPT3 (see Toh et al. (1999)) within Matlab R2014a on an Intel Core i5 CPU running at 2.9 GHz with 4 GB RAM under Windows 7 operating system. The computation time of the robust solution methods is positively correlated with its theoretical complexity. The technique introduced in Section 4.3 is applied to compute  $\mathbf{x}_\mu$  and  $\mathbf{x}_{(\mu,d)}$ . Since the problem size is relative small, all the robust solutions can be obtained within 2s.

**Table 5** The nominal and robust production vectors for  $\sigma = \gamma = 15\%$ . The solutions from the MVE method,  $\mu$  approach,  $(\mu, d)$  approach and RLS method are denoted as  $\mathbf{x}_{MVE}$ ,  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu,d)}$  and  $\mathbf{x}_{RLS}$ , respectively. The solution  $\mathbf{x}_0$  is the nominal solution. The exact ranges of the components of  $\mathbf{x}$  are reported in the last column.

	$\mathbf{x}_0$	$\mathbf{x}_{MVE}$	$\mathbf{x}_\mu$	$\mathbf{x}_{(\mu,d)}$	$\mathbf{x}_{RLS}$	$[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$
AFF	52.28	52.78	51.26	52.12	49.23	[39.75, 67.74]
Manuf.	732.99	741.90	730.17	732.56	731.78	[573.34, 922.36]
Services	505.37	511.67	503.38	505.07	504.97	[386.46, 650.52]
E & H	119.26	119.37	119.05	119.22	119.25	[100.01, 139.25]
Other	406.24	408.00	405.25	406.09	406.11	[330.31, 490.79]
Complexity	Easy	SDP	QP	QP	SOCP	LP
Time (seconds)	0	1.95	0.23	0.84	0.61	—

We again consider the five robustness measures as introduced in §6.1. The numerical results are reported in Table 6. The nominal solution  $\mathbf{x}_0$  is the second best solution with respect to  $Volume, WCD$  and  $MDFS$  and the third best with respect to  $MDW_\mu$  and  $MDW_{(\mu,d)}$ . Note that the robust solution  $\mathbf{x}_{(\mu,d)}$  is very close to  $\mathbf{x}_0$  for small  $\gamma$ . If  $\gamma = 0$ , we have  $\mathbf{x}_{(\mu,d)} = \mathbf{x}_0$ ; if  $\gamma = 1$ , we have  $\mathbf{x}_{(\mu,d)} = \mathbf{x}_\mu$ . The robust solution  $\mathbf{x}_{(\mu,d)}$  has the best (i.e., least)  $MDW_{(\mu,d)}$  and the second best  $MDW_\mu$ . Contrastingly,  $\mathbf{x}_\mu$  has the best  $MDW_\mu$  and the second best  $MDW_{(\mu,d)}$ . For other robustness measures,  $\mathbf{x}_\mu$  is not as good as  $\mathbf{x}_0$  and  $\mathbf{x}_{(\mu,d)}$ . The solution  $\mathbf{x}_\mu$  has the worst (i.e. highest)  $MDFS$ . The robust solution  $\mathbf{x}_{MVE}$  is a centralized solution in the solution set, hence, it has the largest ellipsoid and the best (i.e., least)  $MDFS$ . The solution  $\mathbf{x}_{MVE}$  performs the worst (i.e., highest) with respect to  $MDW_\mu, MDW_{(\mu,d)}$  and  $WCD$ . The robust solution  $\mathbf{x}_{RLS}$  is the most robust solution with respect to  $WCD$ , but geometrically, it is not robust as its corresponding

**Table 6** Numerical result of the robust solution methods for input-output model. The solutions from the MVE method,  $\mu$  approach,  $(\mu, d)$  approach and RLS method are denoted as  $\mathbf{x}_{MVE}$ ,  $\mathbf{x}_\mu$ ,  $\mathbf{x}_{(\mu, d)}$  and  $\mathbf{x}_{RLS}$ , respectively. The solution  $\mathbf{x}_0$  is the nominal solution. The bold numbers show that the corresponding robust solution performs the best (among the nominal and robust solutions) with respect to the corresponding robustness measure.

	$\mathbf{x}_0$	$\mathbf{x}_{MVE}$	$\mathbf{x}_\mu$	$\mathbf{x}_{(\mu, d)}$	$\mathbf{x}_{RLS}$
<i>Volume</i>	42.97	<b>44.27</b>	41.55	42.76	39.57
$MDW_\mu/10^2$	104.22	105.19	<b>104.12</b>	104.15	104.24
$MDW_{(\mu, d)}/10^2$	51.95	52.69	51.94	<b>51.93</b>	51.99
<i>WCD</i>	148.09	155.60	149.99	148.37	<b>148.07</b>
<i>MDFS</i>	107.19	<b>105.45</b>	107.94	107.29	107.51

maximum inscribed ellipsoid has the smallest *Volume*. Moreover, it also has the second worst  $MDW_\mu$ ,  $MDW_{(\mu, d)}$  and *MDFS*.

For other values of  $\sigma$  and  $\gamma$ , or different  $\sigma_{ij}$ 's and  $\gamma_{ij}$ 's for each components of  $A$  and  $\mathbf{b}$ , the above observations remain valid. The observations are also unchanged if we consider squared deviations instead of 2-norm deviations in *WCD* and *MDFS*. For  $\sigma \geq 25\%$ , the RLS solutions are outside the solution set, whereas the solutions  $\mathbf{x}_\mu$  and  $\mathbf{x}_{(\mu, d)}$  remain inside. Since the robust solution  $\mathbf{x}_{MVE}$  is an approximation of the optimal MVE center, we apply the upper bounding method of Section 3.2. The upper bounding *Volume* is 44.38. The obtained lower bound is 44.27, which implies that  $\mathbf{x}_{MVE}$  is very close to the optimal solution. From §6.1 and §6.2, one may observe that the nominal solutions are rather robust against independent interval uncertainties.

### 6.3. Robust Rating for Colley's Matrix Ranking

Colley's bias free college football ranking method was first introduced by Colley (2001). This method became so successful that it is now one of the six computer rankings incorporated in the Bowl Championship Series method of ranking National Collegiate Athletic Association college football teams. The notation here is adapted from Burer (2012).

Colley Matrix Rankings require to solve a system of linear equations  $A\mathbf{x} = \mathbf{b}$ . For  $n$  teams, the  $n \times n$  matrix  $W$  is defined as

$$W_{ij} = \text{number of times team } i \text{ has beaten team } j.$$

In particular,  $W_{ij} = W_{ji} = 0$  if  $i$  has not played against  $j$ , and  $W_{ii} = 0$  for all  $i$ . Note that the  $ij$ -th entry of  $W + W^T$  represents the number of times team  $i$  and team  $j$  has played against each other. Let  $\mathbf{1}$  be the all-ones vector, then the  $i$ -th entry of  $(W + W^T)\mathbf{1}$  and  $(W - W^T)\mathbf{1}$  gives the total number of games played by team  $i$ , i.e., the schedule of the games, and its win-loss spread. The Colley matrix  $A$  and the vector  $\mathbf{b}$  are defined via the schedule of the games and the win-loss spread vector respectively, i.e.,

$$A = 2I + \text{Diag}((W + W^T)\mathbf{1}) - (W + W^T)$$



$$\mathbf{b} = \mathbf{1} + \frac{1}{2}(W - W^T)\mathbf{1},$$

where  $I$  is the identity matrix and  $Diag(\cdot)$  places its vector components into a diagonal matrix. Since the schedule of the games are often predetermined, we only consider uncertainties in the vector  $\mathbf{b}$ . We empirically investigate the robust version of Colley Matrix ratings to modest changes in the win-loss outcomes of inconsequential games. A game is *inconsequential* if it has occurred between two bottom teams, i.e., teams win less than 30% of all the games they played. Suppose  $m$  inconsequential games has been played during the whole season. Let  $\boldsymbol{\zeta} \in \mathbb{R}^m$  denote the perturbation of the games. The game  $j$  switches its outcome if  $\zeta_j = 1$ , and it remains unchanged if  $\zeta_j = 0$ . For all  $j$ , we have  $0 \leq \zeta_j \leq 1$ . Then, we define a matrix  $\Delta \in \mathbb{R}^{n \times m}$ , where

$$\Delta_{ij} = \begin{cases} 1 & \text{if team } i \text{ loses the game } j \\ -1 & \text{if team } i \text{ wins the game } j \\ 0 & \text{otherwise.} \end{cases}$$

The vector  $\Delta\boldsymbol{\zeta}$  represents the possible switches in the outcome of the games. The maximum number of inconsequential games that are allowed to switch their outcomes is less than  $L \in \mathbb{N}^0$ , i.e.,  $\sum_j \zeta_j \leq L$ . The polyhedral solution set is as follows:

$$\text{conv}(\mathcal{X}) = \{\mathbf{x} : A_0\mathbf{x} = \mathbf{b}_0 + \Delta\boldsymbol{\zeta}, \boldsymbol{\zeta} \in \mathcal{U}\},$$

where the matrix  $A_0$  and the vector  $\mathbf{b}_0$  are nominal,  $\text{conv}(\mathcal{X})$  denotes the convex hull of the set  $\mathcal{X}$ , and  $\mathcal{U} = \{\boldsymbol{\zeta} : \boldsymbol{\zeta} \in \{0, 1\}^m, \sum_j \zeta_j \leq L\}$ . The uncertainty set  $\mathcal{U}$  contains all possible integral  $\boldsymbol{\zeta}$ 's (i.e., scenarios). For  $\boldsymbol{\zeta} = \mathbf{0}$ , the nominal rating vector  $\mathbf{x}_0 = A^{-1}\mathbf{b}$  is on the boundary of  $\mathcal{X}$ . Note that the ratings of Colley's Matrix Rankings are not necessarily nonnegative. Since negative ratings are rather rare and their values are marginal (often very close to zero), we restrict ourself to the rating vectors that are nonnegative.

The data we use in this subsection are downloaded from the website Wolfe (2015). The data contains the outcomes of all college football games of 2014. There are  $m = 32$  inconsequential games in total. Same as in Burer (2012), we limit our focus to just games played with  $n = 204$  Football Bowl Subdivision (FBS) teams. Roughly speaking, the FBS includes the largest and most competitive collegiate football programs in the country. We allow at most  $L = 4$  inconsequential games to switch their outcomes. The robust solution  $\mathbf{x}_{MVE}$  is defined as the approximated MVE center of the solution set  $\text{conv}(\mathcal{X})$ . Since the uncertain parameters are dependent (i.e.,  $\sum_j \zeta_j \leq L$ ), the  $\boldsymbol{\mu}$  and  $(\boldsymbol{\mu}, \mathbf{d})$  approaches cannot be applied here. For the polyhedral set  $\text{conv}(\mathcal{X})$ , the RLS method requires solving a 2-norm maximization problem which is NP-hard. Burer (2012) proposes a two-stage method to solve the following MINLP problem:

$$\min_{\mathbf{x}} \max_{\boldsymbol{\zeta} \in \mathcal{U}} \|\mathbf{A}_0\mathbf{x} - \mathbf{b}_0 - \Delta\boldsymbol{\zeta}\|_2.$$

We denote the robust solution of Burer as  $\mathbf{x}_{RLS}$ . Due to the high dimension of the solutions (i.e., 204), we do not report the nominal and robust solutions, and the exact ranges of the components of  $\mathbf{x}$  for this numerical experiment. Since the uncertainty set  $\mathcal{U}$  is discrete, the robustness measures that we consider here are slightly different from those introduced in §6.1:

- *Volume*: the approximated volume of the MVE centered at  $\tilde{\mathbf{x}}$  within  $\text{conv}(\mathcal{X})$
- *MDPS*: the mean 2-norm deviations of  $A_0\tilde{\mathbf{x}}$  from  $\mathbf{b}_0 + \Delta\boldsymbol{\zeta}$  with respect to all possible  $\boldsymbol{\zeta} \in \mathcal{U}$
- *WCD*: the worst-case 2-norm deviations of  $A_0\tilde{\mathbf{x}}$  from  $\mathbf{b}_0 + \Delta\boldsymbol{\zeta}$  with respect to all possible  $\boldsymbol{\zeta} \in \mathcal{U}$  (i.e.,  $\max_{\boldsymbol{\zeta} \in \mathcal{U}} \|A_0\mathbf{x} - \mathbf{b}_0 - \Delta\boldsymbol{\zeta}\|_2$ )
- *MDFS*: the mean 2-norm deviations of uniformly sampled feasible solutions within  $\text{conv}(\mathcal{X})$  (i.e.,  $\mathbb{E}_{\mathbb{P}_{\mathbf{x}}} \|\mathbf{x} - \tilde{\mathbf{x}}\|_2$ , where  $\mathbb{P}_{\mathbf{x}}$  denotes the uniform distribution).

To compute the *MDPS*, we enumerate all the integral  $\boldsymbol{\zeta} \in \mathcal{U}$ . In case of  $m = 32$  inconsequential games with  $L = 4$ , the total number of possible  $\boldsymbol{\zeta}$  is  $\sum_{i=1}^4 \binom{32}{i} = 41,448$ . The *WCD* is the worst-case 2-norm deviations of  $A\mathbf{x}$  from  $\mathbf{b} + \Delta\boldsymbol{\zeta}$  with respect to the 41,448 possible scenarios. The *MDFS* is estimated from the  $10^4$  uniformly sampled solutions in  $\text{conv}(\mathcal{X})$ .

**Table 7** Numerical result of the robust ratings for Colley’s Matrix Ranking. The nominal solution, MVE solution and RLS solution are denoted as  $\mathbf{x}_0$ ,  $\mathbf{x}_{MVE}$  and  $\mathbf{x}_{RLS}$ , respectively. The bold numbers show that  $\mathbf{x}_{MVE}$  performs the best (among  $\mathbf{x}_0$ ,  $\mathbf{x}_{MVE}$  and  $\mathbf{x}_{RLS}$ ) with respect to the corresponding robustness measure.

	$\mathbf{x}_0$	$\mathbf{x}_{MVE}$	$\mathbf{x}_{RLS}$
<i>Volume</i>	0	<b>0.016</b>	0.003
<i>MDPS</i>	2.94	<b>2.68</b>	2.81
<i>WCD</i>	5.48	<b>4.71</b>	4.98
<i>MDFS</i>	1.69	<b>1.16</b>	1.59
Complexity	Easy	SDP	MINLP
Time (seconds)	0	18.89	21.01

From Table 7, it is readily obvious that the solution  $\mathbf{x}_{MVE}$  is the most robust one. Due to the problem definition, the effect of switching the result of the inconsequential games is not symmetric. The solution  $\mathbf{x}_0$  is on the boundary of  $\mathcal{X}$  and it is the least robust solution among all three with respect to the considered measures. These observations still hold if we consider the mean of the sum of squared deviations instead of 2-norm deviations in *MDPS*, *WCD* and *MDFS*. We again evaluate the quality of the approximation  $\mathbf{x}_{MVE}$  by computing its upper bounding volume. The obtained upper bounding volume is 0.030. The optimal volume lies between 0.016 and 0.030. The MINLP problem is solved with CPLEX 12.6 ILOG (2013). The computation times are again in line with the theoretical complexity of the methods. For larger sized problems, one can expect exponential growth in computation time for the RLS method, whereas, the MVE center method remains computationally tractable.

#### 6.4. Robust Article Influence Scores

Around 1996-1998, Larry Page and Sergey Brin, Ph.D. students at Stanford University, developed the PageRank algorithm for rating and ranking the importance of Web pages (see Brin and Page (1999)). An adapted version of PageRank has recently been proposed to rank the importance of scientific journals as a replacement for the traditional impact factor (see Bergstrom et al. (2008)).

Let us consider the following six prestigious journals in the field of Operations Research, i.e., *Management Science* (MS), *Operations Research* (OR), *Mathematical Programming* (MP), *European Journal of Operational Research* (EJOR), *INFORMS Journal on Computing* (IJC) and *Mathematics of Operations Research* (MOR). The journal citation network can be represented as an adjacency matrix  $H$ , where  $H_{ij}$  indicates the number of times that articles published in journal  $j$  during the census period cite articles in journal  $i$  published during the same period. The number of publications for journal  $i$  is denoted as the  $i$ -th component of  $\mathbf{v}$ . We consider the number of citations and publications of the six journals in 2013 and obtain the corresponding  $H$  and  $\mathbf{v}$  from Thomson-Reuters Corp. (2014):

$$H = \begin{matrix} & \begin{matrix} \text{MS} & \text{OR} & \text{MP} & \text{EJOR} & \text{IJC} & \text{MOR} \end{matrix} \\ \begin{matrix} \text{MS} \\ \text{OR} \\ \text{MP} \\ \text{EJOR} \\ \text{IJC} \\ \text{MOR} \end{matrix} & \begin{bmatrix} 607 & 182 & 24 & 542 & 57 & 16 \\ 140 & 317 & 212 & 536 & 97 & 27 \\ 9 & 63 & 375 & 135 & 69 & 25 \\ 20 & 93 & 41 & 2170 & 72 & 2 \\ 2 & 30 & 16 & 75 & 51 & 0 \\ 16 & 58 & 81 & 56 & 0 & 53 \end{bmatrix} \end{matrix} \text{ and } \mathbf{v} = \begin{pmatrix} 165 \\ 96 \\ 123 \\ 469 \\ 58 \\ 38 \end{pmatrix}. \quad (21)$$

There are some modifications that need to be done to  $H$  before the influence vector can be calculated. First, we set the diagonal elements of  $H$  to 0, so that journals do not receive credit for self-citation. Then, we normalize the columns of  $H$ . To do this, we divide each column of  $H$  by its sum. We normalize the vector  $\mathbf{v}$  in the same fashion. The normalized  $H$  and  $\mathbf{v}$  in (21) are as follows:

$$S = \begin{matrix} & \begin{matrix} \text{MS} & \text{OR} & \text{MP} & \text{EJOR} & \text{IJC} & \text{MOR} \end{matrix} \\ \begin{matrix} \text{MS} \\ \text{OR} \\ \text{MP} \\ \text{EJOR} \\ \text{IJC} \\ \text{MOR} \end{matrix} & \begin{bmatrix} 0 & 0.427 & 0.064 & 0.403 & 0.193 & 0.229 \\ 0.749 & 0 & 0.567 & 0.399 & 0.329 & 0.386 \\ 0.048 & 0.148 & 0 & 0.100 & 0.234 & 0.357 \\ 0.107 & 0.218 & 0.110 & 0 & 0.244 & 0.029 \\ 0.011 & 0.070 & 0.043 & 0.056 & 0 & 0 \\ 0.086 & 0.136 & 0.217 & 0.042 & 0 & 0 \end{bmatrix} \end{matrix} \text{ and } \mathbf{w} = \begin{pmatrix} 0.174 \\ 0.101 \\ 0.130 \\ 0.495 \\ 0.061 \\ 0.040 \end{pmatrix}. \quad (22)$$

Finally, we construct the matrix  $A$ , a convex combination of  $S$  and a rank-one matrix, i.e.,

$$A = \alpha S + (1 - \alpha) \frac{1}{n} \mathbf{w} \mathbf{1}^T, \quad 0 \leq \alpha < 1, \quad (23)$$

where  $\alpha$  is the damping factor and  $\mathbf{w} \mathbf{1}^T$  is a  $n \times n$  matrix. The damping factor models the possibility that a searcher choose a random paper out of all papers. Therefore, the closer the  $\alpha$  gets to 1, the

**Table 8** The nominal and robust Article Influence Scores. The nominal solution and MVE solution are denoted  $\mathbf{x}_0$  and  $\mathbf{x}_{MVE}$ , respectively. The exact ranges of the components of  $\mathbf{x}$  and  $AI$  are denoted as  $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$  and  $[\underline{AI}, \overline{AI}]$ , respectively.

	MS	OR	MP	EJOR	IJC	MOR
$\mathbf{x}_0$	0.240	0.338	0.122	0.163	0.043	0.094
$\mathbf{x}_{MVE}$	0.239	0.337	0.121	0.162	0.048	0.093
$[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$	[0.147, 0.336]	[0.257, 0.416]	[0.035, 0.220]	[0.069, 0.259]	[0, 0.142]	[0.016, 0.194]
$AI(\mathbf{x}_0)$	1.424	3.602	0.937	0.255	0.666	2.494
$AI(\mathbf{x}_{MVE})$	1.424	3.598	0.941	0.256	0.663	2.481
$[\underline{AI}, \overline{AI}]$	[1.258, 1.590]	[3.132, 4.076]	[0.737, 1.152]	[0.217, 0.298]	[0.575, 0.756]	[2.039, 2.939]

better the journal's citation structure is represented by the matrix  $A$ . The influence vector  $\mathbf{x}^*$  can be obtained by solving as follows:

$$A\mathbf{x} = \mathbf{x}, \quad \sum_{i=1}^n x_i = 1. \quad (24)$$

From the Perron-Frobenius theorem, we know a unique rating vector  $\mathbf{x}^*$  can be found. The Article Influence score of journal  $i$  can be calculated as follows:

$$AI_i = \frac{S\mathbf{x}^*}{w_i \sum_{i=1}^n (S\mathbf{x}^*)_i}, \quad \forall i.$$

In this subsection, we assume  $\alpha = 90\%$ . Let us consider the matrix  $S$  and vector  $\mathbf{w}$  defined in (22) and denote the obtained matrix in (23) as the nominal matrix  $A_0$ . The nominal influence vector  $\mathbf{x}_0$  is obtained by solving the system of linear equations (24). Since the estimated probabilities are not exact, we take uncertainty in the matrix  $A$  into consideration. Let us consider the following column-wise 1-norm uncertainties in  $A$ :

$$\mathcal{U} = \{\boldsymbol{\zeta} : \|\boldsymbol{\zeta}_j - (\mathbf{a}_0)_{\cdot j}\|_1 \leq \sigma, \boldsymbol{\zeta}_j^T \mathbf{1} = 1, \boldsymbol{\zeta}_j \geq 0, \forall j\},$$

where  $\boldsymbol{\zeta} = [\boldsymbol{\zeta}_1^T \cdots \boldsymbol{\zeta}_n^T]^T \in \mathbb{R}^{n^2}$ ,  $\sigma = 20\%$ ,  $\boldsymbol{\zeta}_j$  and  $(\mathbf{a}_0)_{\cdot j}$  are the  $j$ th column of matrix  $A(\boldsymbol{\zeta})$  and  $A_0$ , respectively. The uncertainties occur in the left-hand side of the system. Note that each column of the nonnegative matrix  $A(\boldsymbol{\zeta})$  is a probability vector, i.e.,  $\boldsymbol{\zeta}_j^T \mathbf{1} = 1$  for all  $j$ . Hence, the uncertain parameters are dependent. The  $\boldsymbol{\mu}$  and  $(\boldsymbol{\mu}, \mathbf{d})$  approaches cannot be applied. Since 2-norm maximization over a polyhedron is an  $NP$ -hard problem, the RLS method is computationally intractable. We consider the tractable (upper-bound) approximation of the RLS proposed in Juditsky and Polyak (2012):

$$(JP) \quad \min_{\mathbf{x}: \sum_i x_i = 1} \|A\mathbf{x} - \mathbf{x}\|_2 + \sigma \sum_{j=1}^n |x_j|.$$

The solution of this approximation coincides with  $\mathbf{x}_0$ . Hence, in the remaining of this section, we do not distinguish the solution of  $(JP)$  from  $\mathbf{x}_0$ . The resulting Article Influence Scores from

**Table 9** Numerical result of the influence vectors. The  $\mathbf{x}_0$  denotes the nominal solution; the  $\mathbf{x}_{MVE}$  is the approximated MVE center obtained by solving (AARP). The bold numbers show that  $\mathbf{x}_{MVE}$  performs the best with respect to all the robustness measures.

	$\mathbf{x}_0$	$\mathbf{x}_{MVE}$
<i>Volume</i>	0.0312	<b>0.0313</b>
<i>MDUP</i>	0.0359	<b>0.0356</b>
<i>MDFS</i>	0.0733	<b>0.0728</b>
Complexity	Easy	SDP
Time (seconds)	0	7.20

the influence vectors are reported in Table 8. The exact ranges of the components of the solution  $\mathbf{x}$  and the AIS  $AI$  are reported. The difference between the nominal and the robust solutions is marginal. The width of  $[\underline{\mathbf{x}}, \overline{\mathbf{x}}]$  and  $[\underline{AI}, \overline{AI}]$  indicates that the system (24) is sensitive to this type of uncertainties.

We again consider the robustness measures *Volume* and *MDFS*. Besides these two measures, the mean 2-norm deviations of  $10^4$  uniformly sampled  $(A, \mathbf{b})$  in  $\mathcal{U}$  are also considered (i.e., *MDUP*). From Table 9, one can observe that the solution  $\mathbf{x}_{MVE}$  is slightly more robust than  $\mathbf{x}_0$  with respect to all three considered measures. In this numerical experiment, the nominal solutions from PageRank-based problems are robust against uncertainties. The obtained upper bounding volume of  $\mathbf{x}_{MVE}$  is 0.0739. The optimal MVE volume lies between 0.0313 and 0.0739. The computation time for  $\mathbf{x}_{MVE}$  is below 10 seconds. We further observe that for a smaller uncertainty  $\sigma$  or damping factor  $\alpha$ , the difference between the nominal solution  $\mathbf{x}_0$  and the robust solution  $\mathbf{x}_{MVE}$  is smaller. In contrast to independent interval uncertainties, from §6.3 and §6.4, we observe that, for column-wise dependent uncertainties, the MVE solutions are more robust than the RLS or nominal solutions. The  $\boldsymbol{\mu}$  or  $(\boldsymbol{\mu}, \mathbf{d})$  approaches cannot be applied if the uncertainties are dependent.

## 7. Conclusion and Future Research

We first generalize the results for interval linear systems. For a system of uncertain linear equations with column-wise uncertainties, we derive a convex representation of the solution set in any orthant. The exact ranges of the components of the solutions can then be determined. We propose two new methods for obtaining robust solutions of systems of uncertain linear equations. We compare the two new methods both theoretically and numerically with the RLS method. The robust solutions from the  $\boldsymbol{\mu}$  or  $(\boldsymbol{\mu}, \mathbf{d})$  approaches and the RLS method may even be outside the solution set. The nominal solutions are very robust against independent interval uncertainties. From the numerical experiments, we observe that, for column-wise dependent uncertainties, the MVE solutions are

more robust than the RLS or nominal solutions. The  $\mu$  or  $(\mu, \mathbf{d})$  approaches cannot be applied if the uncertainties are dependent. Table 10 may be useful for selecting a proper robust solution method for different uncertainty types and robust criteria.

In this paper, we focus on systems with square matrices  $A$ . In principle, our methods can also be applied to non-square matrices and inequalities, but further research is needed. Our results may be useful for other real-life applications, e.g., analysis of mechanical structures, electrical circuit designs and chemical engineering.

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## Appendix. Theoretical comparison of the robust solution methods

Table 10 Theoretical comparison of the robust solution methods.

Robust methods	MVE center ( $x_{MVE}$ )	$\mu$ and $(\mu, d)$ approaches ( $x_\mu$ and $x_{(\mu, d)}$ )	RLS ( $x_{RLS}$ )
Robust solutions	The MVE center of the solution set only with respect to the variable $x$ .	The minimizer of the expected sum of squared deviations with respect to the worst-case distribution.	The minimizer of the worst-case 2-norm deviations.
Mathematical descriptions	$\max_{x, E} \log \det E$ s.t. $x + E\epsilon \in \mathcal{X}, \forall \epsilon: \ \epsilon\ _2 \leq 1$	$\min_x \sup_{\zeta \in \mathcal{P}} \mathbb{E}_{\mathcal{P}_\zeta} (\ A(\zeta)x - b(\zeta)\ _2^2)$	$\min_x \max_{\zeta \in \mathcal{U}} \ A(\zeta)x - b(\zeta)\ _2$
Uncertainty structures	Column-wise, and the components of $A(\zeta)$ and $b(\zeta)$ are affine in $\zeta$ .	The components of $A(\zeta)$ and $b(\zeta)$ are affine in $\zeta$ .	The components of $A(\zeta)$ and $b(\zeta)$ are affine in $\zeta$ .
Uncertainty sets	$\zeta$ resides in polyhedral sets.	A set of probability distributions with known support-including intervals, means and MADs of $\zeta$ .	$\zeta$ resides in independent intervals or ellipsoidal sets.
Optimality	Approximation (lower-bound)	Exact	Exact
Complexities	SDP	QP	SOCp or SDP
Advantages	Geometrically centralized solution. Column-wise dependency in $A(\zeta)$ and $b(\zeta)$ . The solution is feasible and scale invariant.	Minimize the deviations with respect to the worst-case distribution. Allow dependent uncertainties among the columns of $A(\zeta)$ and $b(\zeta)$ . The solution is exact.	Minimize the worst-case 2-norm deviations. Allow dependent uncertainties among the columns of $A(\zeta)$ and $b(\zeta)$ . The solution is exact.
Disadvantages	The solution $x$ is restricted in a specific orthant. Independent uncertainties among the columns of $A(\zeta)$ and $b(\zeta)$ . Not optimizing with respect to the worst-case scenario. The solution is not exact. The solution set has to be bounded.	Independency among the components of $\zeta$ . Only limited statistical information. The solution is sensitive to scaling and may be infeasible.	The choices of the uncertainty sets are limited. The solution is sensitive to scaling and may be infeasible.

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